1 of 1





50 P - $4.60

~~QUALIFIED REQUESTERS MAY OBTAIN COPIES~~
~~OF THIS REPORT FROM DDC.~~



**HYDRONAUTICS, incorporated**
**research in hydrodynamics**

TECHNICAL REPORT 001-12

# DOWNWASH DUE TO A FULLY CAVITATED HYDROFOIL BENEATH A FREE SURFACE

By

B. Yim

July 1963

Prepared Under

Bureau of Ships

Department of the Navy

Contract NObs-78396

# TABLE OF CONTENTS

## LIST OF SYMBOLS

| | |
|---|---|
| 2b | Span of the rear foil |
| $C_L$ | Lift Coefficient $L/(\frac{1}{2}\rho V^2 c)$ in two dimension $L/(\rho V^2 cs)$ in three dimension |
| c | Hydrofoil chord |
| f | Depth of foil below undisturbed free surface |
| g | Acceleration of gravity |
| Im | Imaginary part of |
| $k_6$ | $= g/V^2$ |
| $M_6$ | Strength of source per unit length of span |
| $M_1$ | Nondimensional strength of source |
| N | Strength of doublet |
| Re | Real part of |
| s | Semi-span of front foil |
| V | Velocity at infinity |
| x, y, z | Rectangular cartesian coordinate given as in Figure 1 |
| $\zeta_0, \zeta_1, \zeta_2$ | Angles defined by Equation [15] |
| $\epsilon$ | Nondimensional downwash |
| $\mu$ | Fictitious frictional force |
| $\xi$ | x coordinate of the location of sink related to the length of cavity |
| $\Phi$ | Total velocity potential |
| $\Phi_1, \Phi_2$ | Velocity potential due to vortex and source respectively. |

## LIST OF FIGURES

# INTRODUCTION

Two problems which have become important due to the development of supercavitating hydrofoil boats are considered. One is the form of the cavity produced by a hydrofoil under the free surface; the other is the flow field due to the cavitating hydrofoil under the free surface. These problems are so closely related that they can hardly be separated. However, the former problem has been dealt with, in the linearized version, for the case of zero cavitation number and infinite Froude number (Auslaender, 1962). For the case of finite cavity (non-zero cavitation number), and infinite Froude number, a point drag cavity model has been used to calculate the approximate size of the cavity. Also the approximate three dimensional effect has been discussed (Yim, 1962). The latter problem, without the cavity, has been studied by Wu (1954) Maruo (1953), Nishiyama (1957), Kaplan, Breslin, and Jacobs (1960), and many others. Even without the cavity, this problem is such an immensely difficult one that it has never been fully investigated. The problem is much more difficult with the cavity. Because of this the problems have been treated separately.

It is well known that a body in a uniform fluid flow can be represented by a distribution of singularities such as vortices and sources. When there exists a free surface, singularity distributions suitable for the infinite fluid case must be corrected to take this into account.

~~In this report~~, attention is ~~especially~~ paid to the downwash far downstream of the cavitated hydrofoil. This result should be of use in the estimation of the influence of the forward foil on the rear foil of the hydrofoil craft.

Since the point where the downwash is calculated is far from the cavitated hydrofoil, the effect of the detailed shape of the hydrofoil is small. Therefore, for the sake of simplicity, the cavitated hydrofoil under the free surface is assumed to have the form of a simple model made of a vortex line and a uniform source plus a line sink. Due to the linearity of the potential, the downwash due to the vortex and due to the source may be dealt with separately. For the downwash due to vortex, the recommended expression by Kaplan, Breslin and Jacobs (1960) is used. For the downwash due to the source distribution, the problems are formulated for the cases of both two and three dimensions. Numerical computations made on the IBM 1620 digital computer at HYDRONAUTICS, Incorporated for the downwash at various downstream and spanwise positions at the same depth as the hydrofoil, and covering a large range of Froude numbers, are given in the form of curves.

## FORMULATION OF PROBLEM

A coordinate system is fixed with respect to a hydrofoil which is moving with constant forward velocity V, so that the flow picture would appear to be stationary with a uniform free stream velocity V approaching the hydrofoil. The origin of the right handed rectangular coordinate system O-xyz is located on the mean free surface, the x axis is directed along the free stream, and the z axis is positive upward (See Figure 1). The liquid medium is assumed to be inviscid, incompressible, and homogeneous so that the condition of irrotationality and continuity implies the existence of the velocity potential $\Phi$ which satisfies

$$\nabla^2 \phi = 0 \qquad [1]$$

The boundary condition on the free surface is

$$\frac{\partial^2 \phi}{\partial x^2} + k_o \frac{\partial \phi}{\partial z} + \mu \frac{\partial \phi}{\partial x} = 0 \qquad [2]$$

on $z = 0$, where $k_o = g/V^2$, g denoting the acceleration of gravity and $\mu$ a fictitious frictional force (Lamb, 1945). At infinity

$$\nabla \phi = 0 \qquad [3]$$

Suppose the cavitated hydrofoil is represented by a distribution of vorticity, and sources. Then, the linearity of the governing equation and the boundary conditions for $\phi$ allows us to write

$$\phi = \phi_1 + \phi_2 \qquad [4]$$

where $\phi_1$ is the velocity potential due to the vorticity distribution and $\phi_2$ due to the source distribution.

Since the aft part of the cavity is known to be approximately similar to an ellipsoid (Tulin 1953; Yim, 1962) we may consider the cavitated foil at the depth h, in its simplest form, to be represented by singularities on $z = -h$ as follows. A vortex line, representing the lift due to the foil is located along the span at one quarter chord from the leading edge. Representing the cavity, is a uniform source distribution with strength, $\frac{M_o}{c}$ per unit area extending over the hydrofoil; that is, $(0 \leq x \leq c)$ and $(s \leq y \leq s)$. In addition a sink line is located at $x = \xi$ along the span $(-s \leq y \leq s)$. It must have the same magnitude of strength $M_o$ per unit span as the total source strength along the chord for a unit span to satisfy the closure requirement for the cavity

We first determine $\phi_a$ for the cases of two dimensions and three dimensions separately.

DOWNWASH DUE TO SOURCE DISTRIBUTION

TWO DIMENSIONAL CASE

Since it is more convenient to deal with a doublet distribution in this case we represent the body by doublets, The doublet distribution N equivalent to the source distribution mentioned above 4s

$$\left.\begin{aligned} N &= -2\frac{M_o x}{c} \qquad \text{for } 0 \leq x \leq c \\ N &= -2M_o \qquad \text{for } c \leq x \leq \xi \end{aligned}\right\} \qquad [5]$$

Before determining the downwash at $(x,-f)$ due to this doublet distribution, we first Investigate the downwash due to a single point doublet.

In an infinite medium, the potential due to a point doublet of strength N is

$$\phi_2(x,z) = \frac{N(x-\xi)}{(x-\xi)^2 + (z+f)^2}$$

If we use the integral representation for this (Bateman, 1954)

$$\phi_2(x,z) = \mathrm{Im}\int_0^\infty N \exp[-k(z+f) + ik(x-\xi)]dk \qquad z + f > 0 \quad .$$

To obtain the potential which is produced by a point doublet under the free surface where the boundary condition (2)

$$\frac{\partial^2 \phi_2}{\partial x^2} + k_o \frac{\partial \phi_2}{\partial z} + \mu \frac{\partial \phi_2}{\partial x} = 0 \qquad \text{on } z = 0$$

holds, we put

$$\phi_2 = \mathrm{Im}\int_0^\infty [F \exp(kz + ikx) + N \exp[-k(z+f) + ik(x-\xi)]]dk \quad [6]$$

which certainly satisfies the Laplace equation,

Substituting Equation [6] into [2] we obtain

$$\mathrm{Im}\int_0^\infty Fk\, [k-k_o-i\mu + N(k+k_o-i\mu) \exp(-kf-ik\xi)]\, \exp(ikx)\, dk = 0$$

for any x. Hence we may write

$$F = \frac{-N(k+k_o-i\mu)}{k-k_o-i\mu} \exp(-kf-ik\xi)$$

Hence

$$\phi_2 = \operatorname{Im}\int_0^\infty N\left[\exp\{-k(f+z) + ik(x-\xi)\} - \frac{k+k_o-i\mu}{k-k_o-i\mu}\exp\{-k(f-z) + ik(x-\xi)\}\right]dk$$

$$= N\left\{\frac{(x-\xi)}{(z+f)^2 + (x-\xi)^2} - \frac{x-\xi}{(z-f)^2 + (x-\xi)^2}\right.$$

$$\left. - \operatorname{Im}\int_0^\infty \frac{2k_o \exp\{k(z-f) + ik(x-\xi)\}}{k-k_o-i\mu}\, dk\right\}$$

The downwash due to she point doublet is

$$\frac{\partial\phi_2}{\partial z} = N\left\{\frac{-2(z+f)(x-\xi)}{[(z+f)^2 + (x-\xi)^2]^2} + \frac{2(z-f)(x-\xi)}{[(z-f)^2 + (x-\xi)^2]^2}\right.$$

$$\left. - \operatorname{Im}\int_0^\infty \frac{2k\,k_o \exp\{k(z-f) + ik(x-\xi)\}}{k-k_o-i\mu}\, dk\right\} \qquad [7]$$

By substition of Equation [5] in the above Equation [7], the downwash due to a given doublet distribution [5] is obtained as follows,

$$\frac{\partial\phi_2}{\partial z} = \frac{2M_o}{c}\left[\int_0^c \frac{+\ 4f(x-\xi)\xi}{\{(z+f)^2 + (x-\xi)^2\}^2}\, d\xi + Im\int_0^c\int_0^\infty \frac{2kk_o e^{k(z-f)+ik(x-\xi)}\xi}{k-k_o-i\mu} dkd\xi\right.$$

$$\left. + c\int_c^{\xi_o} \frac{4f(x-\xi)d\xi}{\{(z+f)^2+(x-\xi)^2\}^2} + Im\int_c^{\xi_o}\int_0^\infty \frac{2kk_o e^{k(z-f)+ik(x-\xi)}c}{k-k_o-i\mu} dkd\xi\right]$$

$$= \frac{2M_o}{c}\left[-\int_0^c \frac{2f\ d\xi}{4f^2 + (x-\xi)^2} + \frac{2f\ c}{4f^2 + (x-\xi_o)^2}\right.$$

$$\left. - Im\int_0^c\int_0^\infty \frac{12\ k_o\ e^{k(z-f)+ik(x-\xi)}}{(k-k_o-i\mu)}\, dkd\xi + Im\int_0^\infty \frac{12k_o ce^{-2kf+ik(x-\xi_o)}}{k-k_o-i\mu}\, dk\right]$$

Carrying out the contour integration similar to that indicated in Appendix I,

$$\frac{\partial\phi}{\partial z} = \frac{2M_o}{c}\left[\tan^{-1}\frac{x-c}{2f} \quad an \quad \frac{}{2f} + \frac{2f\ c}{4f^2 + (x-\xi_o)^2}\right.$$

$$- 4\ c\pi k_o\ e^{-2k_o f}\sin k_o(x-\xi_o)$$

$$+ 4\pi\ \exp(-2k_o f)(\cos k_o(x-c)-\cos k_o x)$$

$$-\int_0^\infty \frac{2\exp\{-m(x-\xi_o)\}\, k_o(k_o\cos 2mf+m\sin 2mf)c}{m^2 + k_o^2} +$$

$$\left.\int_0^\infty\int_0^c \frac{2\exp\{-m(x-\xi)\}(k_o\cos 2mf+m\sin mf)k_o}{m^2 + k_o^2}\, d\xi dm\right] \qquad [8]$$

The last two integrals on the right hand side of the above equation represent local disturbances which die down rapidly with increasing distance x downstream and become negligible at values of x of interest. Hence they may be neglected in the present investigation. Values of downwash obtained from the resulting equation are shown plotted along with the three dimensional case in Figures 4 - 9.

THREE DIMENSIONAL CASE

In a manner similar to that; used in two dimensional case the velocity potential due to a given source distribution can be obtained.

In an infinite medium, the potential $\Phi_{21}$ due to a source distribution M(a) per unit area is

$$\Phi_{21} = \int_a \frac{M(a)\, da}{\sqrt{\{(x-\xi)^2 + (y-\eta)^2 + (z-f)^2\}}}$$

$$= \frac{1}{2\pi} \int_a \int_{-\pi}^{\pi} \int_0^{\infty} M(a) \exp[ikw-k(f-z)]\, dk d\theta da$$

where $f-z \geq 0$, $w = (x-\xi)\cos\theta + (y-\eta)\sin\theta$ and da represents a surface element of the basic plane z = -f and the real part of the integral representation is taken.

As in the two dimensional case, from the boundary condition (2) on the free surface x = 0, the velocity potential may be written as

$$\phi_2 = - Vx + \phi_{21} + \phi_{22}$$

$$= - Vx + \frac{1}{2\pi} \int_a \int_{-\pi}^{\pi} \int_0^{\infty} M(a) \Big[ \exp\{ikw-k(f+z)\}$$

$$-\exp\{ikw-k(f-z)\} - \frac{2k_o \sec^2\theta \, \exp\{ikw-(f-z)\}}{k-k_o \sec^2\theta - i\mu\sec\theta} \Big] dk d\theta da \qquad [9]$$

for $f + z \geq 0$ and $f - z \geq 0$.

Now if $M(a)$ is taken for our model as

$$\left. \begin{aligned} M &= \frac{M_o}{c} \text{ per unit area in } 0 \leq x \leq c \\ &\qquad\qquad -s \leq y \leq s \\ M &= M_o \text{ per unit length on } x = \xi_o \\ &\qquad\qquad -s \leq y \leq s \end{aligned} \right\} \qquad [10]$$

Then, the downwash at the point $(x,0,-f)$ may be written as

$$\frac{\partial \Phi_2}{\partial z} = -\frac{M_o}{c}\int_{-s}^{s}\int_{o}^{c} 2f\{(x-\xi)^2 + \eta^2 + 4f^2\}^{-\frac{3}{2}}\, d\xi d\eta$$

$$-\frac{M_o}{c\pi}\int_{-s}^{s}\int_{o}^{c}\int_{-\pi}^{\pi}\int_{o}^{\infty} \frac{kk_o\sec^2\theta\ \exp(ikw_o-2kf)}{k-k_o\sec^2\theta-i\mu\ \sec\ \theta}\, dkd\theta d\xi d\eta$$

$$+ M_o\int_{-s}^{s} 2f\{x-\xi_1)^2 + \eta^2 + 4f^2\}^{-\frac{3}{2}}\, d\eta$$

$$+\frac{M_o}{\pi}\int_{-s}^{s}\int_{-\pi}^{\pi}\int_{o}^{\infty} \frac{kk_o\sec^2\theta\ \exp(ikw_1-2kf)}{k-k_o\sec^2\theta-i\mu\ \sec\ \theta}\, dkd\theta d\eta \qquad [11]$$

where

$$w_a = (x-\xi)\cos\theta-\eta\sin\theta$$

$$w_1 = (x-\xi_1)\cos\theta-\eta\sin\theta$$

All the orders of' integrations can be changed here although they look quite different because of the complicated singularity (See Appendix II).

As in the two dimensional case the 2nd and the 4th integral can be simplified by the use of the contour integration (See Appendix I) i.e.,

$$I = -8\pi \int_{-\pi/2}^{\pi/2} k_o \sec^2\theta \csc\theta \exp(-2fk_o \sec^2\theta) \cos[(x-\xi_1)\cos\theta - s\sin\theta] k_o \sec^2\theta$$

$$+ 8\pi \int_0^s \int_{\frac{\pi}{2}-\zeta_o}^{\pi/2} k_o^2 \sec^4\theta \exp(-2fk_o \sec^2\theta) \sin[(x-\xi_1)\cos\theta - \eta\sin\theta] k_o \sec^2\theta] d\theta$$

$$+ 4\int_0^s \int_{-\pi/2-\zeta_o}^{\pi/2-\zeta_o} \int_0^\infty \frac{\exp[-(x-\xi_1)m\cos\theta + \eta m\sin\theta]}{k_o^2 \sec^4\theta + m^2} mk_o \sec^2\theta \Big[ k_o \sec^2\theta \cos(2mf)$$

$$+ m\sin(2mf)\Big] dm\,d\theta\,d\eta \qquad [12.1]$$

where $\zeta_o = \text{Arctan}\,[\eta/(x-\xi_1)]$

For the change of the order of Integration in the 2nd term for example, Figure 3 should be noted. The domain of the integration is the shaded area in the figure. Thus,

$$I = -16\pi \int_{0}^{\pi/2} k_o \sec^2 \operatorname{cosec}\theta \, \exp(-2fk_o \sec^2\theta) \sin[(x-\xi_1) k_o \sec\theta] \sin[s k_o \sin\theta \sec^2\theta]$$

$$+ 8\pi \int_{\pi/2-\zeta_1}^{\pi/2} k_o \sec^2\theta \, \operatorname{cosec}\theta \, \exp(-2fk_o \sec^2\theta)[\cos\{k_o(x-\xi_1)\sec\theta - sk_o \sin 8 \sec^2\theta\} - 1] \, d\theta$$

$$+ 4 \int_{-\pi/2-\zeta_1}^{\pi/2-\zeta_1} \int_{0}^{\infty} \frac{\exp[-(x-\xi_1) m\cos\theta + \eta m \sin\theta \}}{(k_o{}^2 \sec^4\theta + m^2)\sin\theta} k_o \sec^2\theta \{k_o \sec^2\theta \cos(2mf) + m\sin(2mf)\} \, dm \, d\theta$$

$$+ 8 \int_{-\pi/2-\zeta_1}^{\pi/2} \int_{0}^{\infty} \frac{k_o \sec^2\theta \{k_o \sec^2\theta \cos(2mf) + m\sin(2mf)\}}{\sin\theta (k_o^2 \sec^4\theta + m^2)} \, dm \, d\theta \qquad [13]$$

where

$$\zeta_1 = \operatorname{Arctan}\{s/(x-\xi_1)\}$$

However, since

$$\int_0^\infty \frac{\cos(2mf)}{k_o^2 \sec^4\theta + m^2}\, dm = \frac{\pi \exp(-2fk_o\sec^2\theta)}{2\, k_o\sec^2\theta}$$

$$\int_0^\infty \frac{m \sin(2mf)}{k_o^2 \sec^4\theta + m^2}\, dm = \frac{\pi}{2} \exp(-2fk_o\sec^2\theta) \quad \text{(See Bateman, 1954)}$$

$$I = -16\pi \int_0^{\pi/2} k_o\sec^2\theta \ \text{cosec}\,\theta \exp(-2fk_o\sec^2\theta)\sin[(x-\xi_1)k_o\sec\theta]\ \sin[sk_o\sin\theta\sec^2\theta]$$

$$+ 8\pi \int_{\pi/2-\zeta_1}^{\pi/2} k_o\sec^2\theta \ \text{cosec}\,\theta \exp(-2fk_o\sec^2\theta)\ \cos\{k_o(x-\xi_1)\sec\theta - sk_o \sin\theta\sec^2\theta\}d\theta$$

$$+ \int_{-\pi/2-\zeta_1}^{\pi/2-\zeta_1} \int_0^\infty \frac{\exp[-(x-\xi_1)m\cos\theta + sm\sin\theta]}{(k_o{}^2\sec^4\theta + m^2)\sin\theta} k_o\sec^2\theta\{k_o\sec^2\theta\cos(2mf) + m\sin(2mf)\}dm\,d\theta \quad [14]$$

Hence, for the distribution of singularities given by Equation [10], the downwash at (x,0-f) is

$$\frac{\partial\Phi_2}{\partial z} = \frac{2M_o}{c}\left[-\,\text{Arctan}\frac{xs}{2f\sqrt{(x^2+s^2+4f^2)}} + \text{Arctan}\frac{(x-c)s}{2f\sqrt{\{(x-c)^2+s^2+4f^2\}}}\right]$$

$$+ \frac{4M_o fs}{\{(x-\xi_1)^2+4f^2\}\sqrt{\{(x-\xi_1)^2+s^2+4f^2\}}}$$

$$-16\,M_o\int_0^{\pi/2}\exp(-2k_o f\sec^2\theta)\sin(k_o s\sin\theta\sec^2\theta)\Big\langle[\sec^2\theta\sin\{k_o(x-\xi_1)\text{se}$$

$$-\frac{\sec\theta}{c}\,[\cos\{(x-c)k_o\sec\theta\} - \cos(xk_o\sec\theta)]\Big\rangle\text{cosec}\theta d\theta$$

$$+ 8M_o\int_{\pi/2-\zeta_1}^{\pi/2} k_o\sec^2\theta\,\text{cosec}\,\theta\,\exp(-2k_o f\sec^2\theta)\cos[k_o\sec^2\theta\{(x-\xi_1)\cos\theta-s\sin$$

$$+ \frac{8M_o}{c}\int_{\pi/2-\zeta_o}^{\pi/2}\exp(-2k_o f\sec^2)\sec\theta\,\text{cosec}\,\theta\Big\{\sin[k_o\sec^2\theta\{(x-c)\cos\theta-s\sin\theta\}$$

$$-\sin[k_o\sec^2\theta(x\cos\theta-s\sin\theta)]\Big\}\,d\theta$$

$$+\frac{8M_o}{c}\int_{\pi/2-\zeta_2}^{\pi/2-\zeta_o}\exp(-2k_o f\sec^2\theta)\sec\theta\,\text{cosec}\theta\sin[k_o\sec^2\theta\{(x-c)\cos\theta-s\sin\theta\}]d$$

[15]

where

$$\zeta_0 = \text{Arctan}(s/x), \zeta_1 = \text{Arctan}[s/(x-\xi_1)] \text{ and } \zeta_2 = \text{Arctan}[s/(x-c)]$$ [15]

and the local effect given by the last term of Equation [14], which dies down rapidly when x is large is neglected.

In Equation [15] the integrands of all the integrals are oscillating functions whose frequencies become faster and amplitudes smaller for increasing $\theta$. Furthermore, because of the factor cosec $\theta$ exp$(-2k_o f \sec^2\theta)$, the major contribution to the integral occurs near $\theta = 0$, when x, or $x-\xi_1$ is larger than s

$$\frac{\pi}{2} - \text{Arctan}\ [s/(x-\xi_1)] > \frac{\pi}{4}\ .$$

Hence, the last three integrals in Equation [15] may well be neglected.

For computation, the variable of integration is changed by

$$\tan\theta = t$$

or

$$\sec^2\theta = t^2 + 1 \quad d\theta = \frac{dt}{t^2+1}$$ [16]

Then the first integral of Equation [15] becomes,

$$16M_o \int_0^\infty \exp[-2k_o f(t^2+1)] \frac{\sin\{k_o st\sqrt{(t^2+1)}\}}{\sqrt{(t^2+1)}} \Big\langle k_o(t^2+1)\sin\{k_o(x-\xi_1)\sqrt{(t^2+1)}\}$$

$$-\frac{\sqrt{(t^2+1)}}{c}[\cos\{(x-c)k_o\sqrt{(t^2+1)}\} - \cos\{xk_o\sqrt{(t^2+1)}\}]\Big\rangle dt \qquad [17]$$

If $y \neq 0$, considering $y-\eta$ instead of $-\eta$ in Equations [12] and [12.1] (also see Appendix I), we can write the terms corresponding to the first four terms in Equation [15]

$$\frac{\partial\phi_2}{\partial z} = -\Bigg[\frac{M_o}{c}\{\text{Arctan}\frac{xt}{2f\sqrt{(x^2+t^2+4f^2)}} - \text{Arctan}\frac{(x-c)t}{2f\sqrt{[(x-c)^2+t^2+4f^2]}}$$

$$-\frac{2M_o ft}{[(x-\xi_1)^2+4f^2]\sqrt{[(x-\xi_1)^2+t^2+4f^2]}}\Bigg]_{t=y-s}^{t=y+s}$$

$$-16M_o \int_0^{\pi/2} \exp(-2k_o f\sec^2\theta) \frac{\sin(k_o s\sin\theta\sec^2\theta)}{\sin\theta} \Big\langle k_o\sec^2\theta\sin\{k_o(x-\xi_1)\sec\theta\}$$

$$-\frac{\sec\theta}{c}[\cos\{(x-c)k_o\sec\theta\} - \cos(xk_o\sec\theta)]\Big\rangle \cos(k_o y\sin\theta\sec^2\theta)d\theta$$

[18]

The average of the downwash over a near tandem foil of the same span as the forward foil is given approximately by

$$\frac{1}{2s}\int_{-s}^{s}\frac{\partial\phi_2}{\partial z}\,dy = \frac{M_o}{2s(x^2+4f^2)}\{-4f\sqrt{(x^2+4s^2+4f^2)}+4f\sqrt{(x^2+4f^2)}\}$$

$$+\frac{2M_o}{2s\{(x-\xi_1)^2+4f^2\}}\{2f\sqrt{(x-\xi_1)^2+4s^2+4f^2}-2f\sqrt{\{(x-\xi_1)^2+4f^2\}}$$

$$+\frac{16M_o}{2s}\int_0^{\pi/2}\exp(-2k_o f\sec^2\theta)\left[\frac{\cos\{2k_o s\ \sin\theta\sec^2\theta\}-1}{k_o\sin^2\theta\sec^2\theta}\right]$$

$$\times\Big\langle[k_o\sec^2\theta\sin\{k_o(x-\xi_1)\sec\theta\}$$

$$-\frac{\sec\theta}{c}[\cos\{(x-c)k_o\sec\theta\}-\cos(xk_o\sec\theta)]\Big\rangle d\theta \qquad [19]$$

where the first two terms in Equation [18] are approximated by the first order terms of their Taylor's expansion. By the change of variable given by Equation [16], the last integral of Equation [19] becomes

$$\frac{16M_o}{2s}\int_0^\infty \exp[-2k_o f(1+t^2)]\frac{1}{k_o t^2(1+t^2)}\,[\cos(2k_o st\sqrt{(1+t^2)})-1]$$

$$X \left< [k_o(1+t^2)\sin(k_o(x-\xi_1)\sqrt{(1+t^2)})] \right.$$

$$\left. -\frac{\sqrt{(1+t^2)}}{c}\,[\cos((x-c)k_o\sqrt{(1+t^2)})-\cos(xk_o\sqrt{(1+t^2)})]\right> dt$$

The numerical computation is performed for each full cycle of the integrand until the magnitude of the integral for the cycle becomes very small. All the integrals are added. These computations were carried out on the IBM 1620 digital computer at HYDRONAUTICS, Incorporated.

By nondimensionalizing $\epsilon = \frac{\partial\phi}{\partial z}/V$, and $M_1 = 2M_o/(Vc)$ the quantity $\epsilon/M_1$ at the points $(x,y,-f)$ and that averaged over the rear foil are plotted in Figures 4-14.

## LIFTING CASE

### DOWNWASH DUE TO VORTEX LINE

In two dimensions the downwash at point $(x,z)$ due to a lifting line located at $(0,f)$ is [Kaplan, Breslin and Jacobs, 1960]

$$\frac{\epsilon}{C_L} = \frac{cx}{4\pi}\left[\frac{1}{x^2+(z+f)^2} + \frac{1}{x^2+(z-f)^2}\right] + k_o c e^{k_o(z-f)} \cos k_o x$$

when we neglect the local effect.

In three dimensions, we adopt Kaplan's recommended expression for the downwash at (x,z) due to lifting line from (0,-b,-f) to (0,b-f) with an elliptical distribution of circulation, averaged over the span of the rear foil 2b.

$$\frac{\epsilon}{C_L} = \frac{c}{\pi bs}\left\langle b-\{[(f+z)^2+s^2-b^2]^2+4(f+z)^2b^2\}^{\frac{1}{4}}\sin\beta'\right\rangle - \frac{c}{\pi s}\left\{1-\frac{(f-z)/s}{\left[1+\frac{(f-z)^2}{s}\right]^{\frac{1}{2}}}\right.$$

$$+\frac{4c}{\pi b}\int_0^{\pi/2} e^{k_o(z-f)\sec^2\theta}\,\frac{\sec\theta}{\sin^2\theta}\,J_1(k_o s\sec^2\theta\sin\theta)\sin(k_o b\sec^2\theta\sin\theta)\cos(k_o x\sec\theta)\,d\theta$$

where $J_1$ is the Bessel function of order 1 and,

$$\sin\beta' = \left[\frac{1}{2}\left\{1-\frac{(f+z)^2+s^2-b^2}{\left[\{(f+z)^2+s^2-b^2\}+4(f+z)^2b^2\right]^{\frac{1}{2}}}\right\}\right]^{\frac{1}{2}}$$

The downwash due to lifting foils is plotted in Figures 15-17 for various depths and aspect ratios. The span of the rear foil is the same as that of the front foil for all cases shown.

# DISCUSSION

The downwash due to the source distribution or the cavity depends on many parameters; Froude number, the points of interest, the size of the cavity, the depth of submergence of the foil, and the length of the span. Figures 4 - 10 show the downwash on the centerline y = 0. In Figure 9 the oscillatory nature of the downwash along the centerline with distance downstream far Froude numbers 2-5 is shown for the case f/c = 1, $\xi_1 = 2$, and aspect ratio = 6. As is well known, the wave length is approximately proportional to the square of the Froude number $\frac{V}{\sqrt{gf}}$ . For large Froude numbers (> 10), the magnitude of the downwash is small for the parameter ranges $x/c \geq 15$ and $2 \leq \xi_1 \leq 8$ which are considered in this report (Figure 4-6).

In the medium Froude number range ($1 < F_r \leq 4$) the variation of the downwash is very large for the same range of parameters considered. The effect of depth is not only in decreasing the Froude number far the same speed, but also in the decrease of the magnitude of downwash for the same Froude number, Although the case of small Froude number (less than 2) is not calculated here, it is easy to see that the downwash will be small for $F_r < 1$ because of the factor $\exp(-2fk_o\sec^2\theta)$ in the integrand of Equation [18]. This implies also that at large depths of submergence the downwash becomes small. Of course as we approach infinite depth, the limiting value of the downwash approaches zero for our model of source distributions located at the same depth as the foil.

However, increasing the depth while holding the speed constant may cause the Froude number to pass through a range such as to produce increasing downwash before it begins to approach to zero due to large depth. Since the foil is operating with a cavity the Froude number M may be expected to be quite large. Also, it is true that large Froude numbers produce large cavities. Therefore, although the nondimensional downwash $\epsilon/M_1$ for a large Froude number is small, the magnitude of $\epsilon$ may be large since $M_1$ increases with the size of cavity. $M_1$ can be estimated approximately from the size of the point drag cavity model (Yim 1962) and for the case of small cavitation number (Auslaender, April 1962).

The effect of span is shown in Figures 8 and 10. In Figure 8, the smooth variation of the downwash from aspect ratio 4 to $\infty$ can readily be seen. Since in each case in Figures 4-7 ( AR = 6) the corresponding two dimensional case (AR = $\infty$) is shown, Figure 8 may be of help in the estimation of the downwash for different aspect ratio for the parameters of Figures 4-7. In Figure 10, the variation of the downwash along the spanwise direction for many Froude numbers is shown. As the Froude number is increased the spanwise variation becomes smaller.

Figures 11-14 show the downwash averaged along a rear foil which has the same span as the front foil. The downwash variations are not very different from those along the centerline.

Figures 15-17 show the downwash due to the vortex line, The parameters of interest are taken exactly the same as for the case of the source distribution. The significant difference from the case of the source distribution is that the downwash becomes large when the Froude number becomes large for the three dimensional case. The variation of downwash for AR = 4 to AR = ∞ foils is shown for each Froude number in Figure 17.

The following relations lead to an order of magnitude relation between $C_L$ and $M_1$ for practical hydrofoils. It can be written approximately that

$$\frac{M_1}{2} \approx \frac{\alpha}{2\pi}$$

where $\alpha = \frac{\partial z}{\partial \pi}$, the slope of cavity, also

$$C_L/\delta \geq \frac{\pi}{2}$$

where $\delta$ is the angle of attack, in two dimensions under a free surface and zero cavitation number. Therefore, if $\alpha$ is about five times as large as $\delta$, the graphs of downwash due to both the source and the vortex in this report; could be read with the same scale.

## APPENDIX I

$$I = \int_{-S}^{S} \int_{-\pi}^{\pi} \int_{0}^{\infty} \frac{kk_o \sec^2\theta \, \exp(ikw_1 - 2kf)}{k - k_o \sec^2\theta - i\mu\sec\theta} \, dk\,d\theta\,d\eta$$

$$w_1 = (x - \xi_1) \cos\theta - \eta \sin\theta$$

$$= A \cos(\theta + \zeta_o)$$

In order for the integration with respect to k to be performed, at should be paid to figures of $\cos(\theta+\zeta)$ and $\cos\theta$, since the contours of gration are different depending on the sign of $w_1$ as shown in Figure 2. the signs of $\cos(\theta+\zeta_1)$ and $\cos\theta$ are opposite the contribution from the residue at the singularity

$$k = k_o \sec^2\theta + i\mu \sec\theta$$

is null because the singularity is outside of the contour.

# APPENDIX I

$$I = \int_{-s}^{s} \int_{-\pi}^{\pi} \int^{\infty} \frac{k k_o \sec^2\theta \exp(ikw_1 - 2kf)}{k - k_o \sec^2\theta - i\mu\sec\theta} \, dk d\theta d\eta$$

$$w_1 = (x - \xi_1) \cos\theta - \eta \sin\theta$$

$$= A \cos(\theta + \zeta_o)$$

In order for the integration with respect to k to be performed, attention should be paid to figures of $\cos(\theta+\zeta)$ and $\cos\theta$, since the contours of integration are different depending on the sign of $w_1$ as shown in Figure 2. When the signs of $\cos(\theta+\zeta_1)$ and $\cos\theta$ are opposite the contribution from the residue at the singularity

$$k = k_o \sec^2\theta + i\mu \sec\theta$$

is null because the singularity is outside of the contour.

$$I = 2\int_0^s 2\pi i\left[\int_{-\pi/2}^{\pi/2-\zeta_0} - \int_{-\pi}^{-\pi/2-\zeta_0} - \int_{\pi/2}^{\pi}\right] k_0^2\sec^4\theta\exp[ik_0\sec^2\theta\{(x-\xi_1)\cos\theta-\eta\sin\theta\}-2fk_0\sec^2\theta]\,dm\,d\theta\,d\eta$$

$$+ 2\int_0^s\int_{-\pi/2-\zeta_0}^{\pi/2-\zeta_0}\int_0^\infty \frac{mk_0\sec^2\theta(k_0\sec^2\theta+im)}{k_0^2\sec^4\theta+m^2}\exp[-m\{(x-\xi_1)\cos\theta-\eta\sin\theta\}-2imf]\,dm\,d\theta\,d\eta$$

$$+ 2\int_0^s\int_{-\pi/2-\zeta_0}^{\pi/2-\zeta_0}\int_0^\infty \frac{mk_0\sec^2\theta(k_0\sec^2\theta-im)}{k_0^2\sec^4\theta+m^2}\exp[-m\{(x-\xi_1)\cos\theta-\eta\sin\theta\}+2imf]\,dm\,d\theta\,d\eta$$

where

$$\zeta_0 = \text{Arctan}\,\frac{\eta}{x-\xi_1}$$

Hence,

$$I = -8\pi \int_{0}^{s} \int_{-\pi/2}^{\pi/2-\zeta_o} k_o^{2} \sec^{4}\theta \; e^{-2fk_o \sec^{2}\theta} \sin[\{(x-\xi_1)\cos\theta - \eta\sin\theta\}k_o \sec^{2}\theta]\, d\theta$$

$$+ 4 \int_{0}^{s} \int_{-\pi/2-\zeta_o}^{\pi/2-\zeta_o} \int_{0}^{\infty} \frac{m k_o \sec^{2}\theta}{m^{2}+k_o^{2}\sec^{4}\theta} \exp\{-m\{(x-\xi_1)\cos\theta - \eta\sin\theta\}\} \{k_o \sec^{2}\theta \cos(2mf)$$

$$+ m \sin(2mf)\}\, dm\, d\theta\, d\eta$$

## APPENDIX II

The change of the order of integrations of the integral (13) with respect to $\eta$ and the other variables may be written as follows.

$$I = \int_{-\pi}^{\pi}\int_{0}^{\infty}\int_{-s}^{s} \frac{kk_o\sec^2\theta\ \exp(-k[i\{(x-\xi_1)\cos\theta-\eta\sin\theta\}-2f]}{k-k_o\sec^2\theta-i\mu\sec\theta}\, d\eta\, dk\, d\theta$$

$$= -\int_{-\pi}^{\pi}\int_{0}^{\infty} \frac{2k_o\sec^2\theta\ \exp(-k[i\{(x-\xi_1)\cos\theta-s\ \sin\theta\}-2f]}{i\ \sin\theta(k-k_o\sec^2\theta-i\mu\sec\theta)}\, dk\, d\theta,$$



By the result of Appendix I,

$$I = -8\pi\int_{-\pi/2}^{\pi/2-\zeta_1} k_o\sec^2\theta\ \exp(-2fk_o\sec^2\theta)\sin[\{(x-\xi_1)\cos\theta-s\ \sin\theta\}k_o\sec^2\theta]d\theta$$

$$+ 4\int_{-\pi/2-\zeta_1}^{\pi/2-\zeta_1}\int_{0}^{\infty} \frac{k_o\sec^2\theta\exp[-m\{(x-\xi_1)\cos\theta-s\ \sin\theta\}]}{(k_o^2\sec^4\theta + m^2)\sin\theta}\{k_o\sec^2\theta\cos(2\sigma f)+m\sin(2mf)\}dm\,d\theta$$

where

$$\zeta_1 = \text{Arctan}\,[s/(x-\xi_1)]$$

From this, exactly the same result as Equation [14] can be easily obtained. The other cases of changing the order of integrations can also be easily shown to be valid.

Auslaender, J., "The Linearized Theory for Supercavitating Hydrofoils Operating at High Speeds Near a Free Surface," Journal of Ship Research Volume 6, No. 2 (October 1962).

Auslaender, J., "Low Drag Supercavitating Hydrofoil Sections," HYDRONAUTICS, Incorporated Technical Report 001-7 (April 1962).

Bateman, H., "Tables of Integral Transform Vol. 1" Edited by Erdélyi, A., McGraw-Hill Book Company (1954).

Kaplan, P., Breslin, J. P., and Jacobs, W. R., "Evaluation of the Theory for Flow Pattern of a Hydrofoil of Finite Span," Journal of Ship Research, Vol. 3, No. 4 (March 1960).

Lamb, Sir Horace "Hydrodynamics," Sixth Edition, Dover Publication (1945).

Maruo, H., "The Effect of the Surface of Water on a Submerged Wing," J.S.N.A. Japan 86, pp 43-55 (February 1953) (In Japanese).

Nishiyama, T., "Study of Submerged Hydrofoil," 60th Anniversary Series Vol. 2., J.S.N.A. Japan (1957).

Tulin, M. P., "Steady Two-Dimensional Cavity Flow About Slender Bodies," DTMB Report 834 (1953).

Wu, Y. T., "A Theory for Hydrofoils of Finite Span," J. of Math. and Phys. Vol. 33 No. 3 (October 1954).

Yim, B., "Finite Length Cavities Beneath Free Surface," HYDRONAUTICS, Incorporated Technical Report 119-4 (1962).

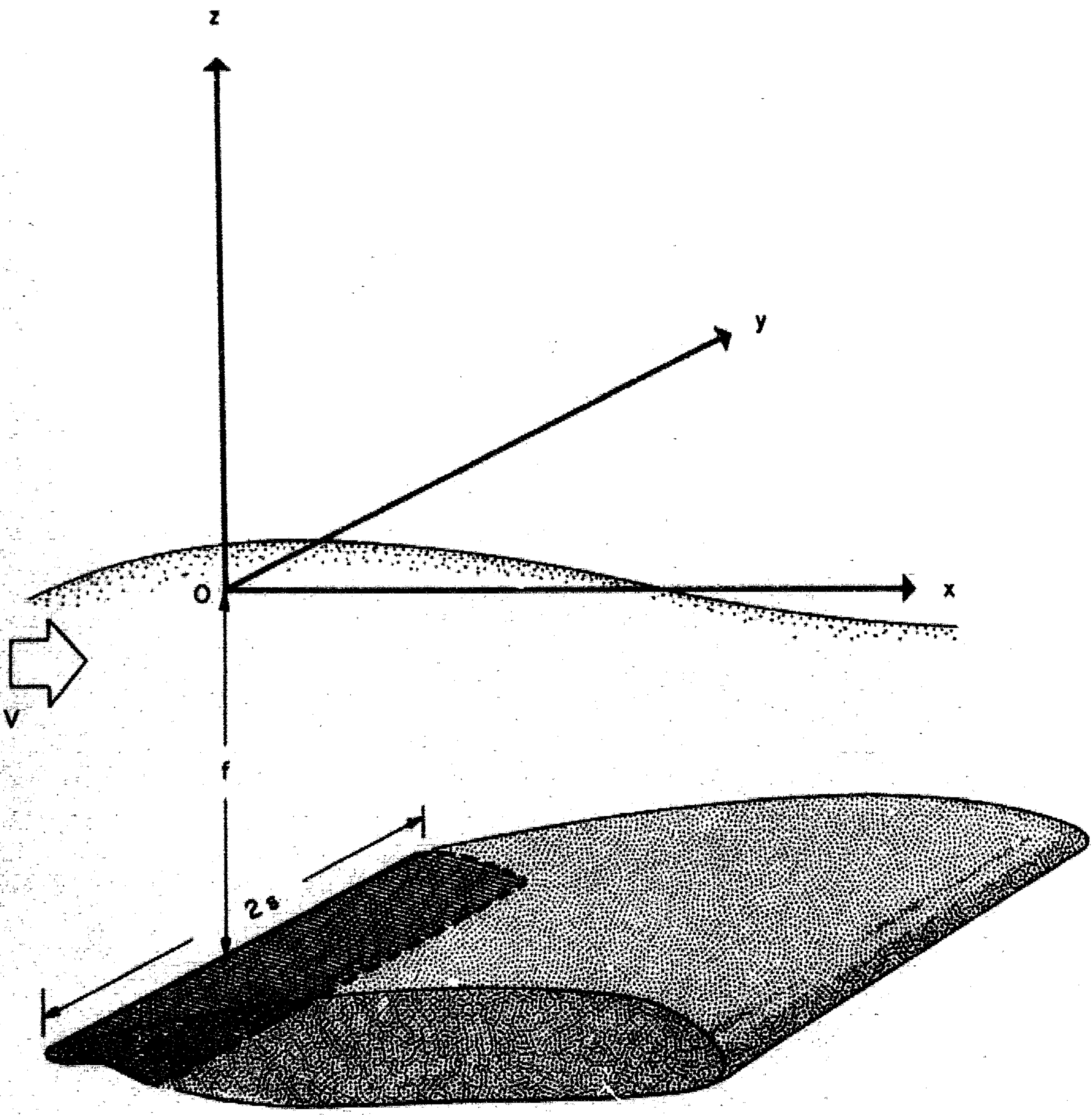
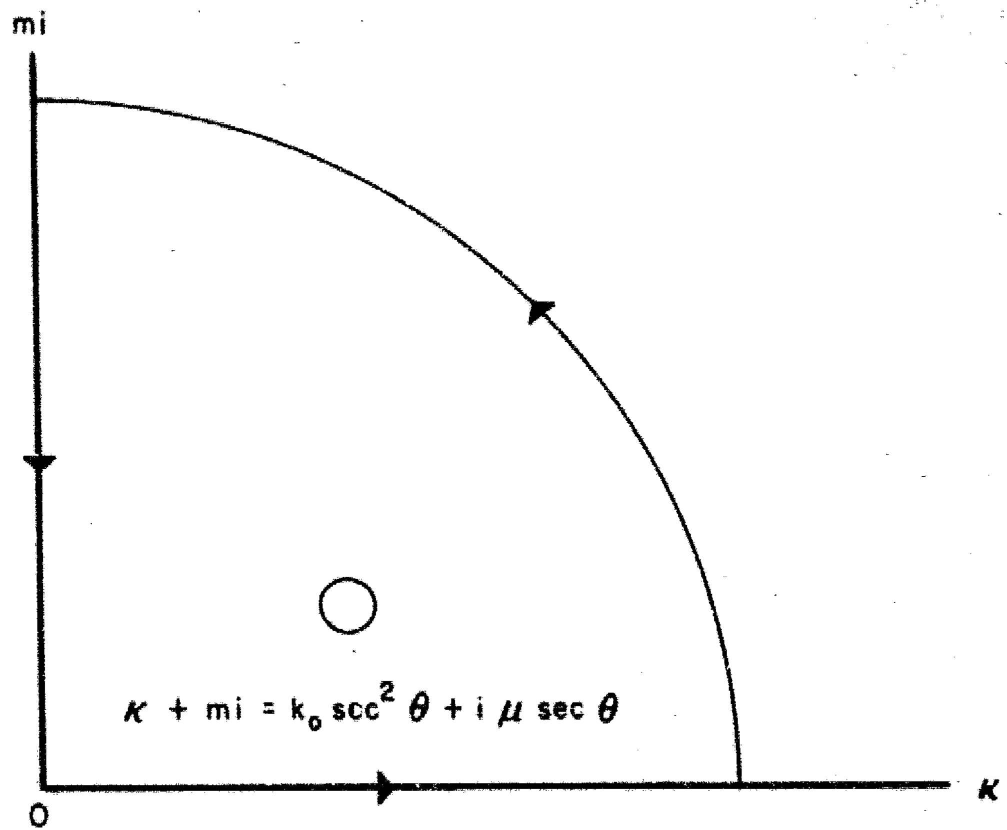
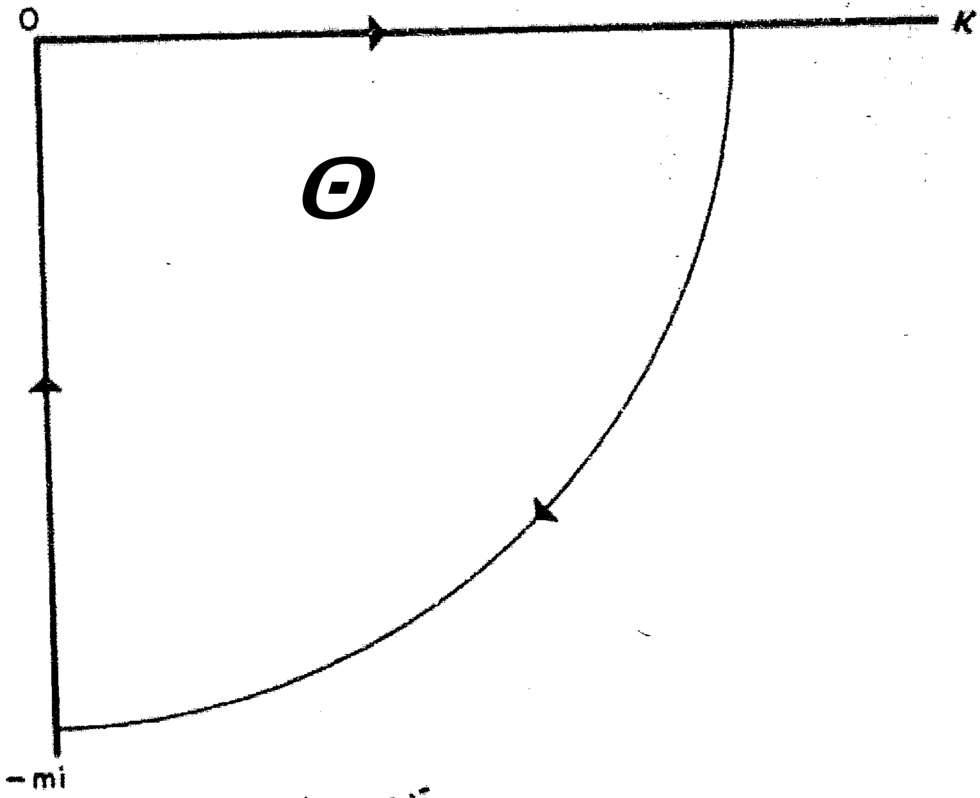



FIGURE 1 - COORDINATE SYSTEM FOR THE MOTION OF CAVITATED HYDROFOIL UNDER FREE SURFACE




WHEN $\cos(\theta - \zeta_1) > 0$




WHEN $\cos(\theta - \zeta_1) < 0$

FIGURE 2 – CONTOURS OF INTEGRATION FOR INTEGRAL (12)

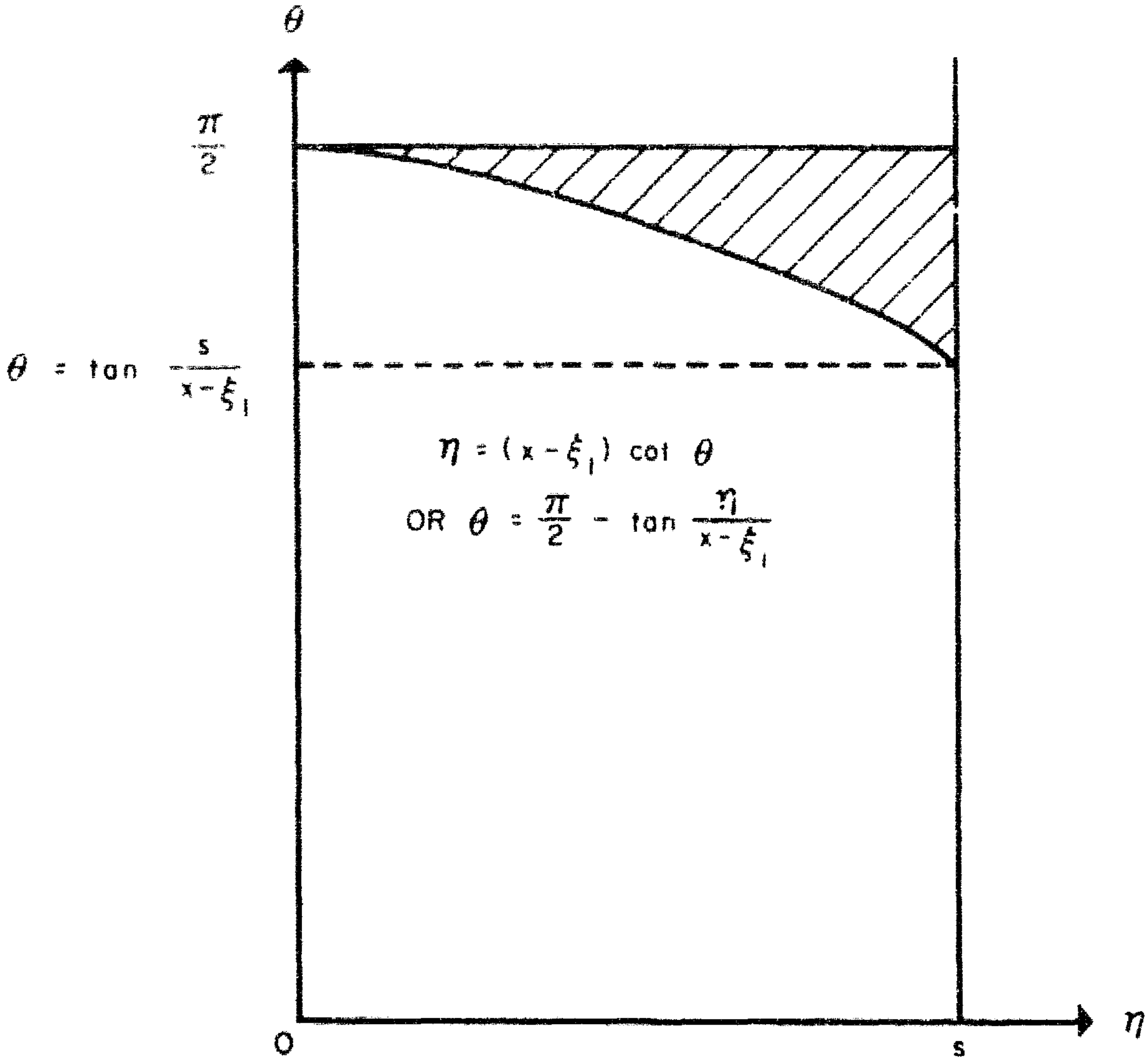



FIGURE 3 — DOMAIN OF INTEGRATION FOR AN INTEGRAL IN EQUATION (13)

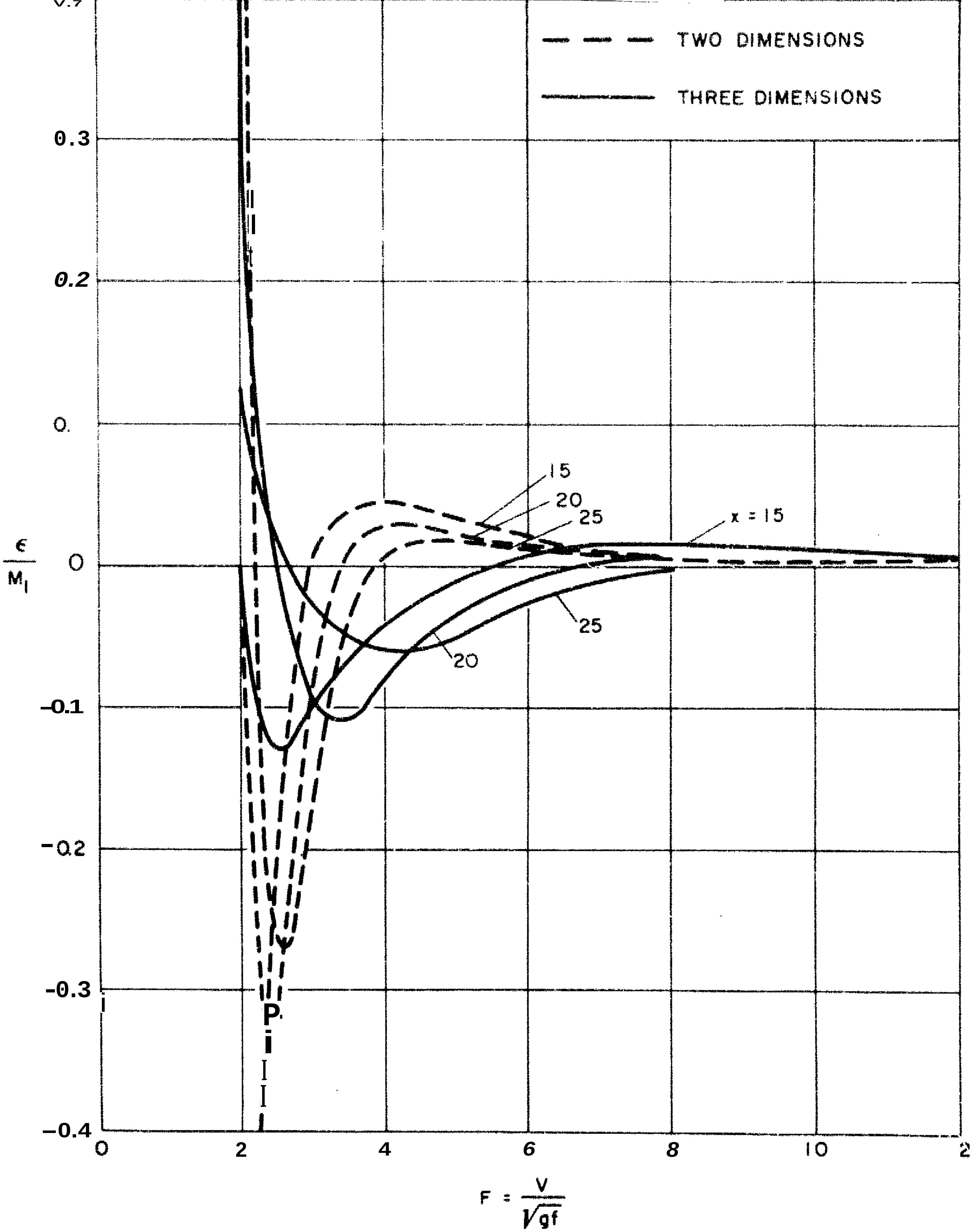
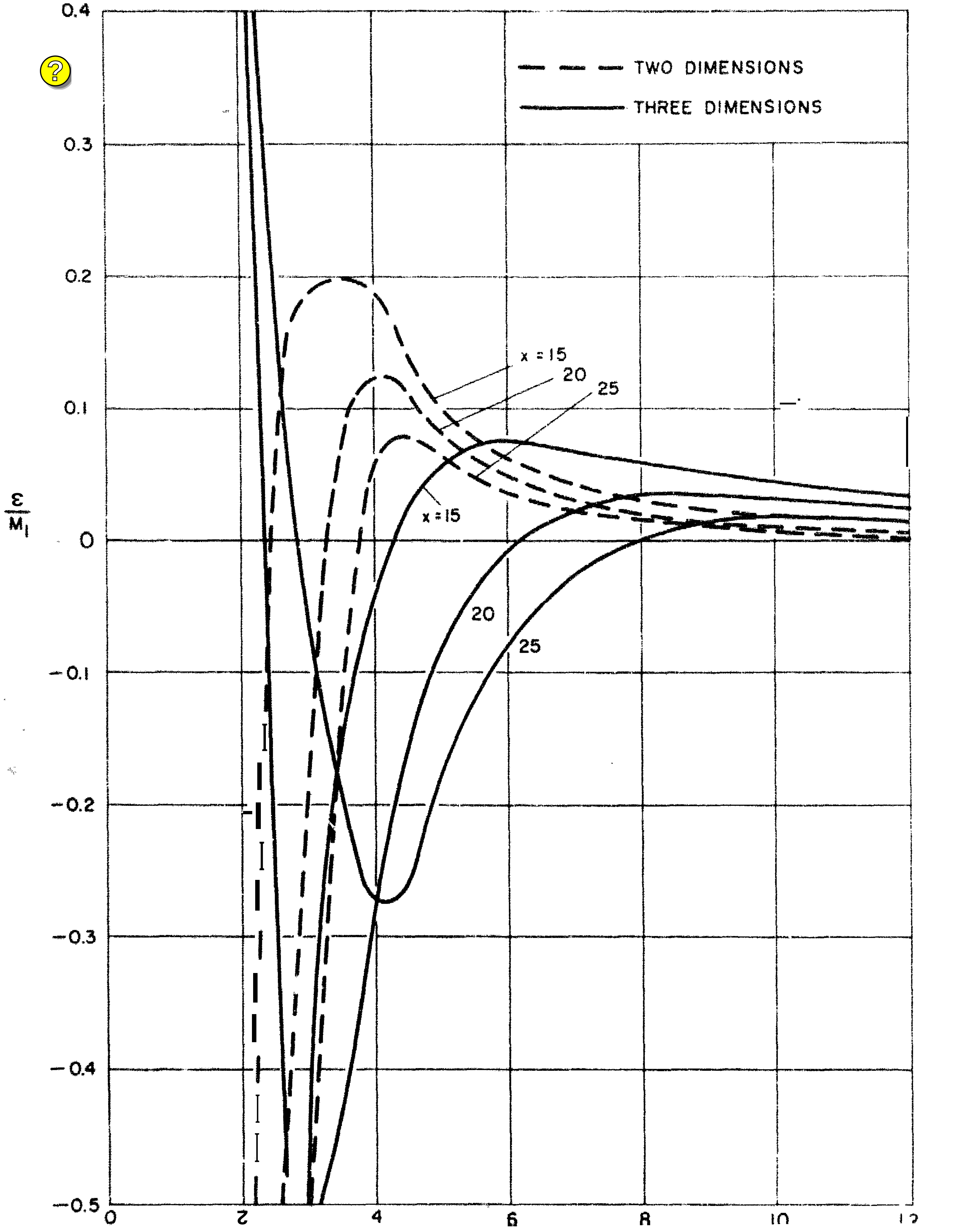



FIGURE 4 — DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$A\!R = 3,\ f/c = 1,\ y/c = 0,\ \xi_1 = 2$


TWO DIMENSIONS
THREE DIMENSIONS
x = 15
20
25
x = 15
20
25
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
ε/M₁
0
2
4
6
8
10
12

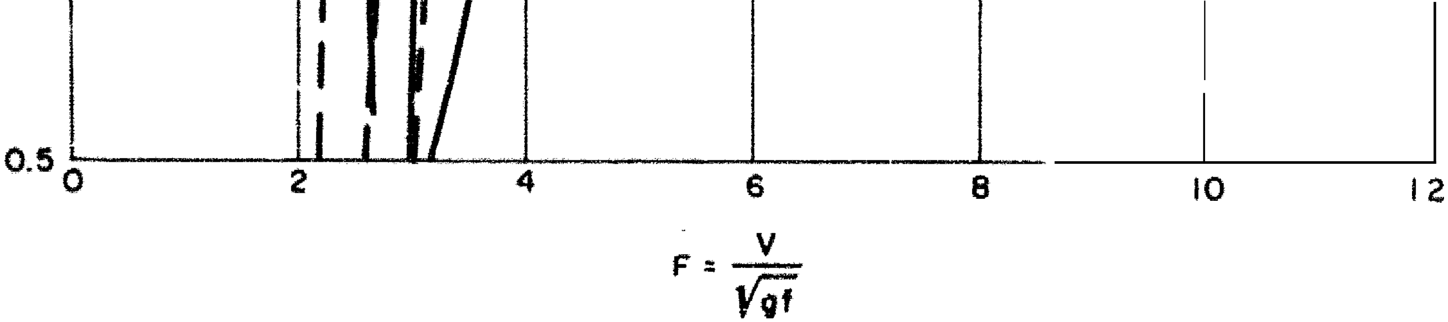



FIGURE 5 – DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$AR = 6,\ f/c = 1,\ y/c = 0,\ \xi_1 = 6$

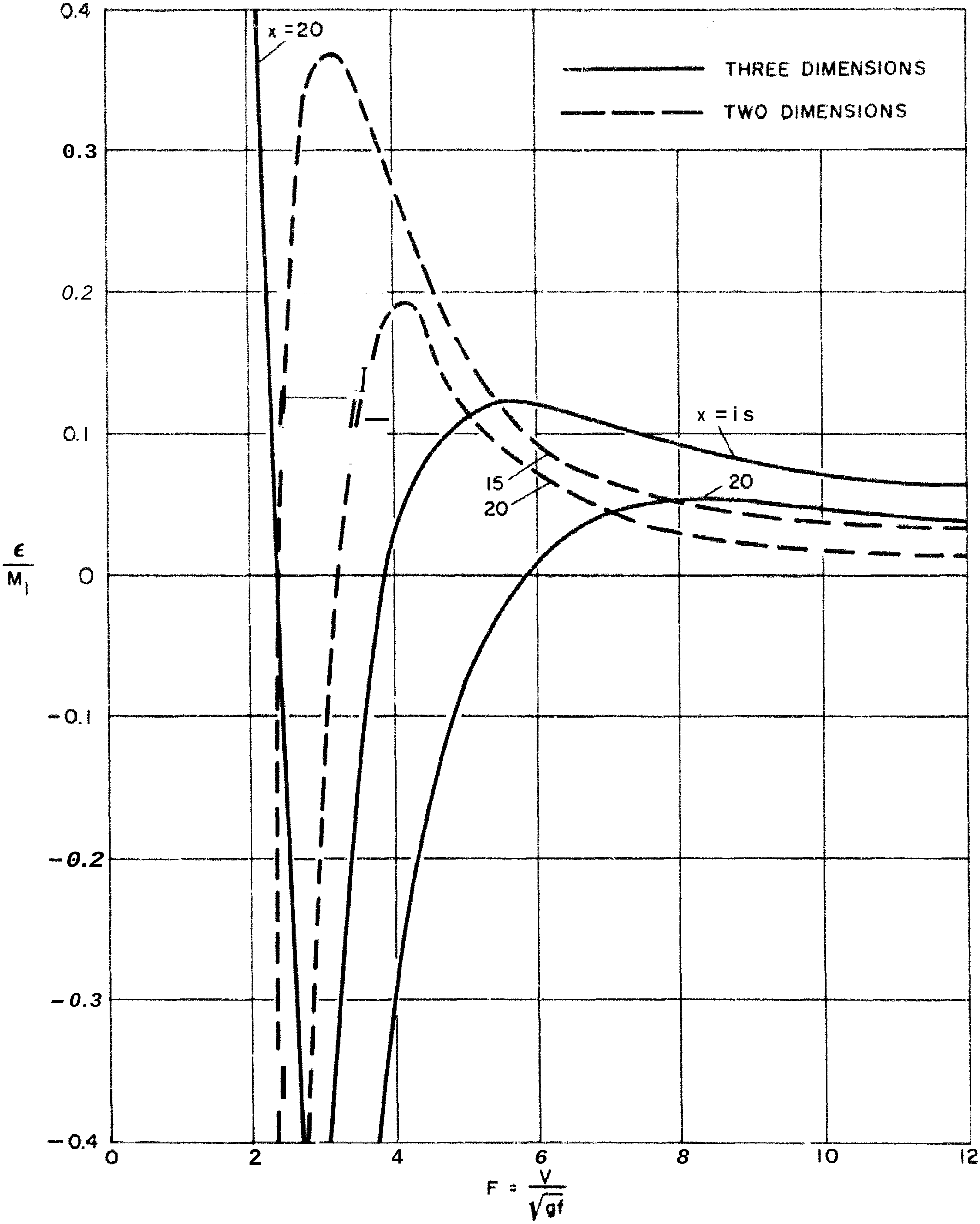



FIGURE 6 - DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$AR = 3,\ f/c = 1,\ y/c = 0,\ \xi_1 = 8$

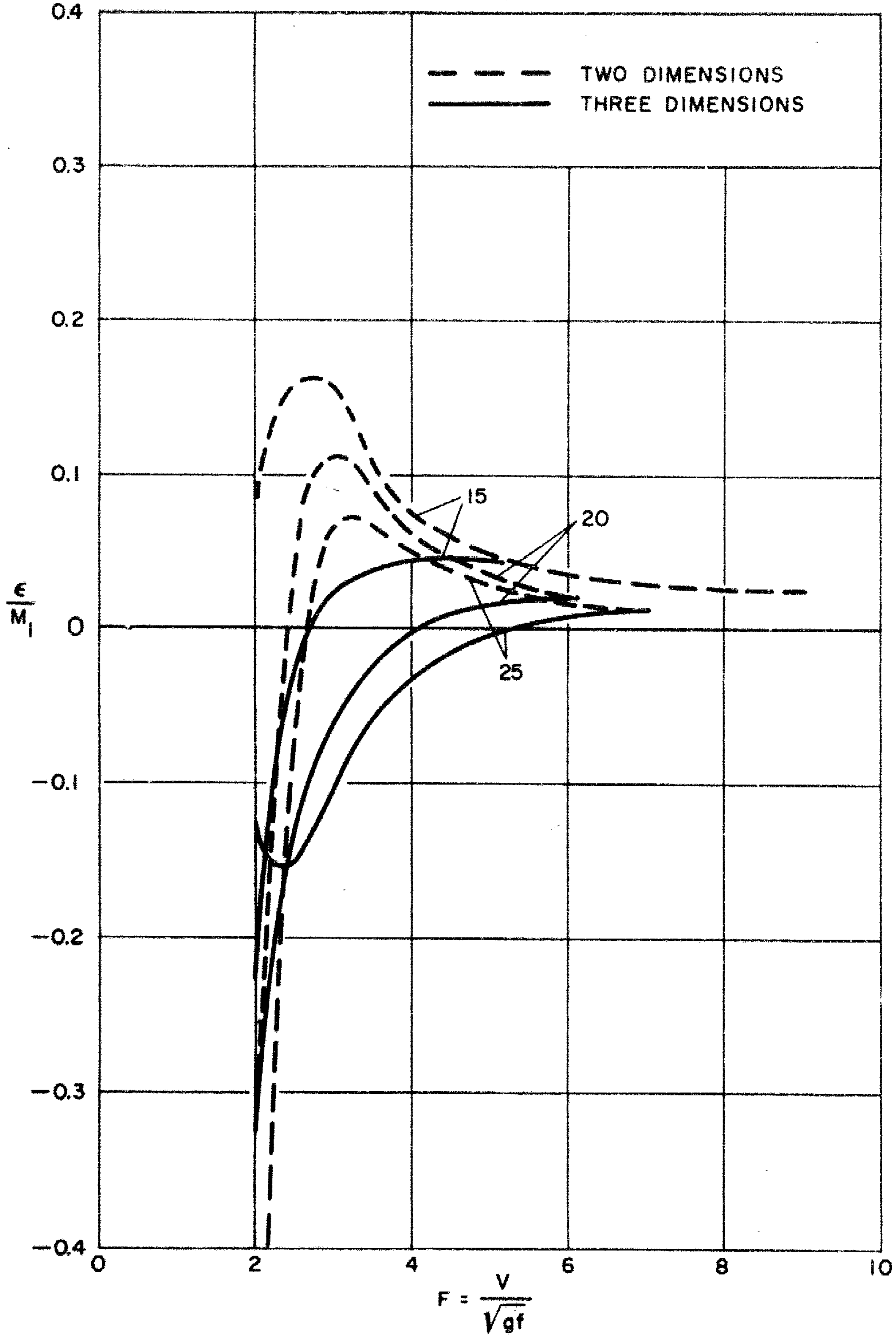
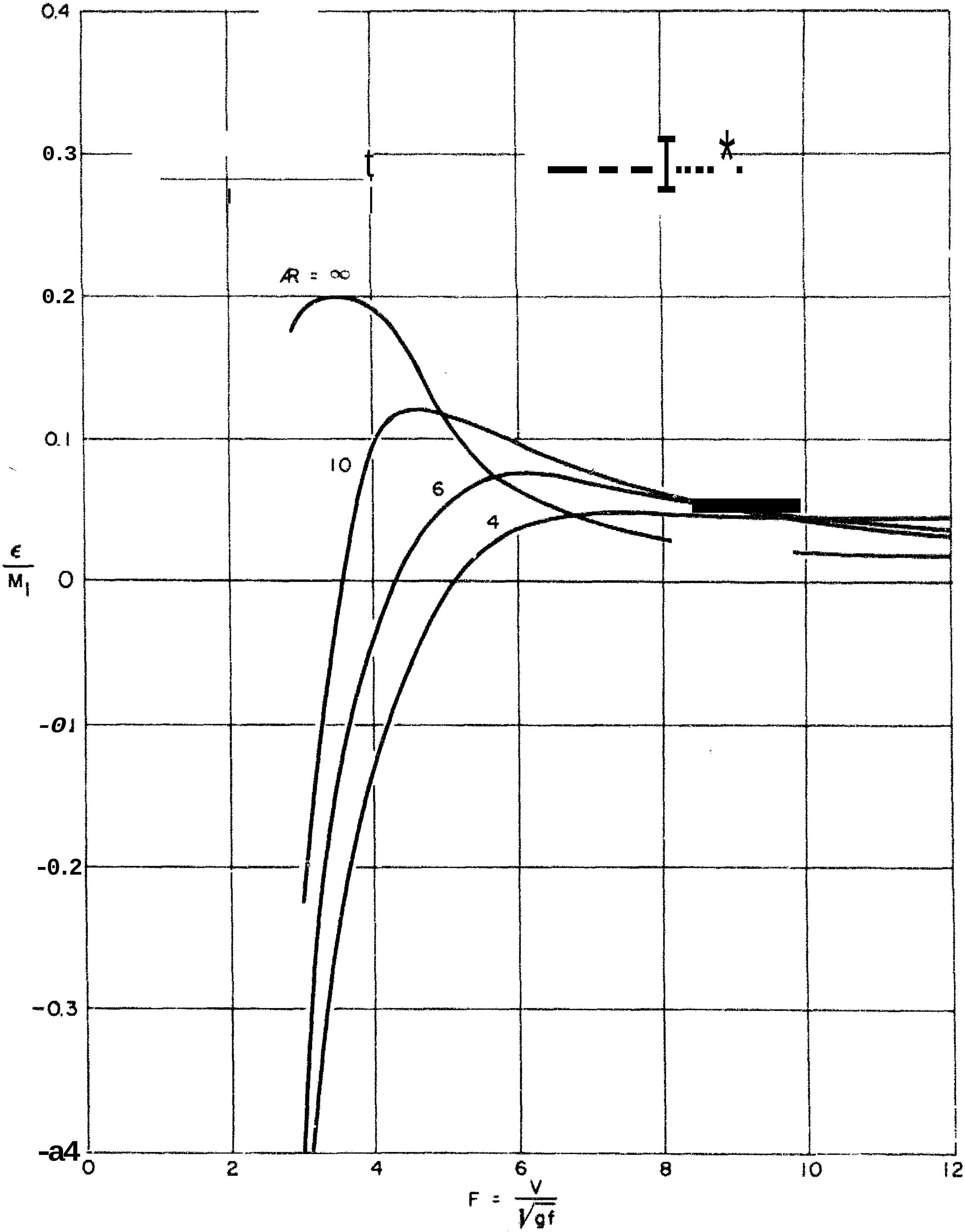



FIGURE 7 — DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$AR = 6,\ f/c = 2,\ y/c = 0,\ \xi_1 = 6$



0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
2
4
6
8
10
12
$\frac{\epsilon}{M_1}$
$F = \frac{V}{\sqrt{gf}}$
AR = ∞
10
6
4

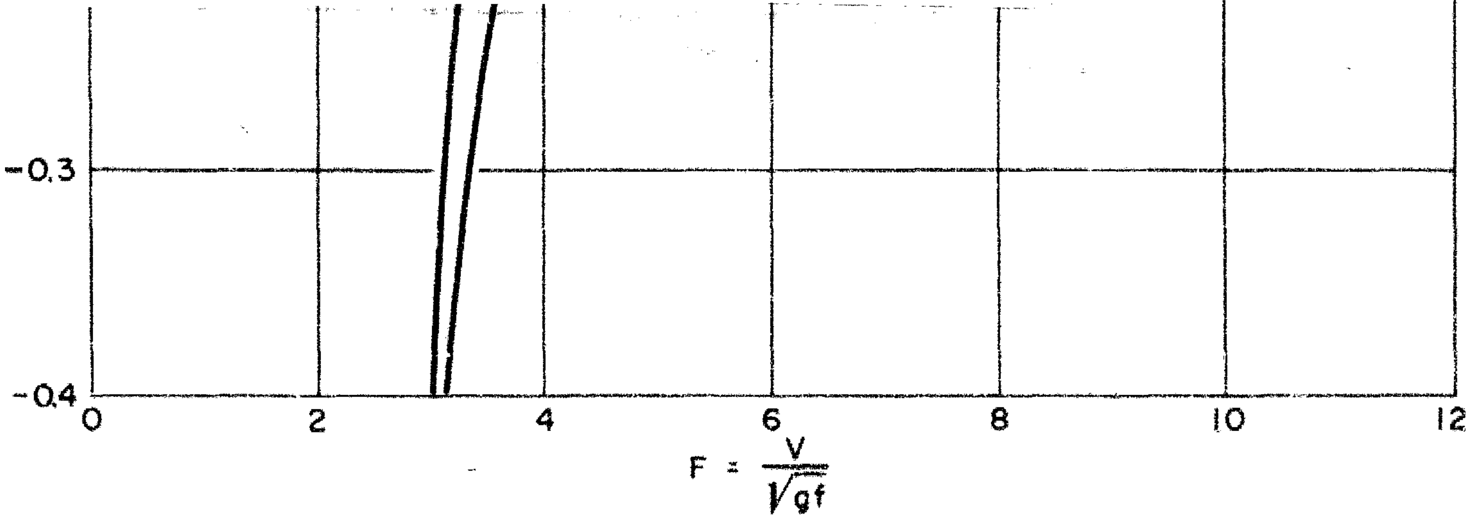
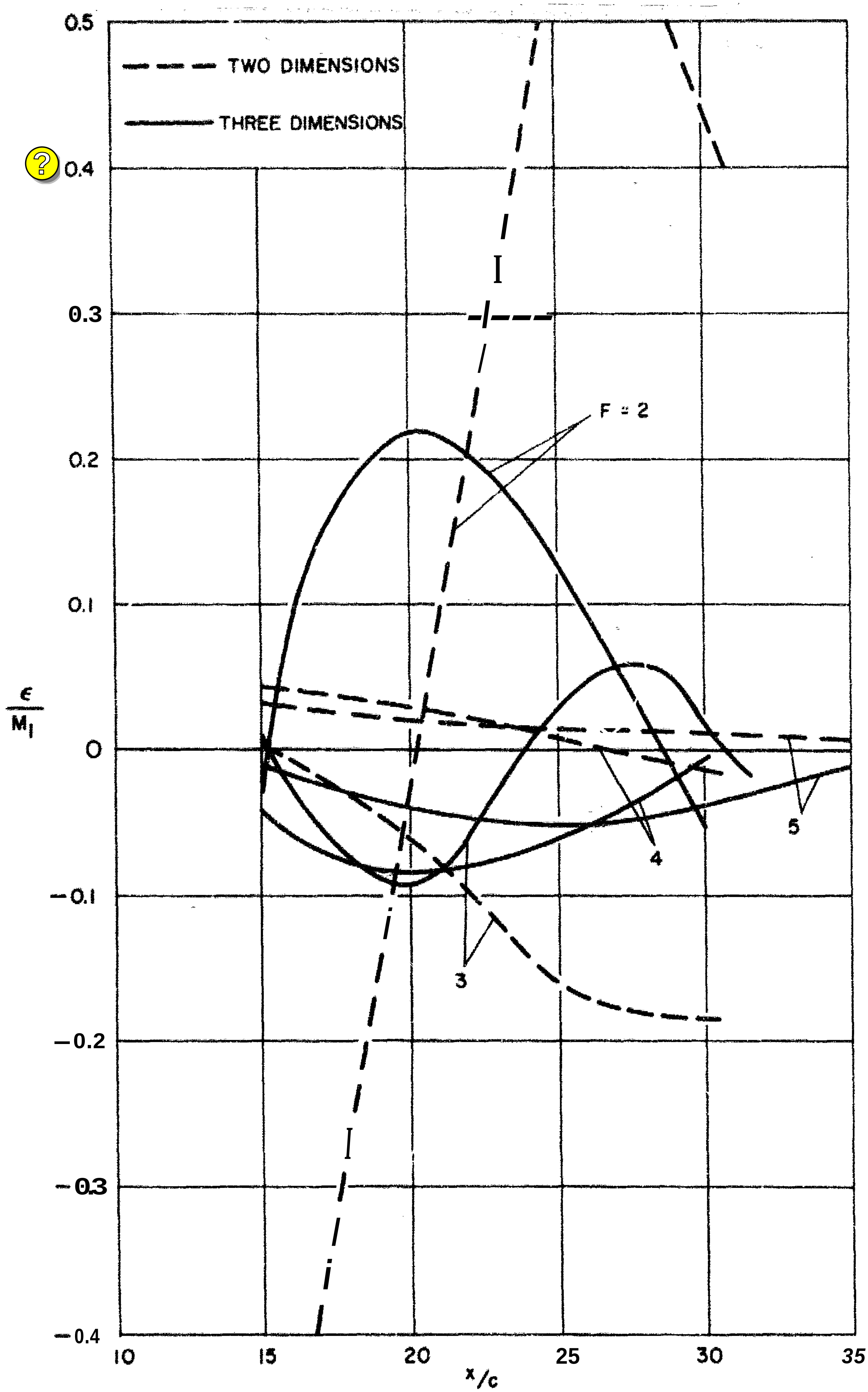



FIGURE 8 – DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$f/c = 1,\ y/c = 0,\ x/c = 15,\ \xi_1 = 6$



TWO DIMENSIONS
THREE DIMENSIONS
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
$\frac{\epsilon}{M_1}$
I
F = 2
3
4
5
10
15
20
25
30
35
x/c

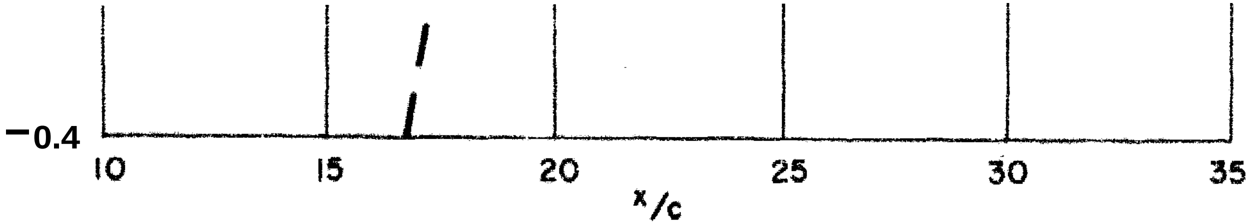



FIGURE 9 — DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$AR = 6,\ f/c = 1,\ y/c = 0,\ F = V/\sqrt{gf}\ ,\ \xi_1 = 2$

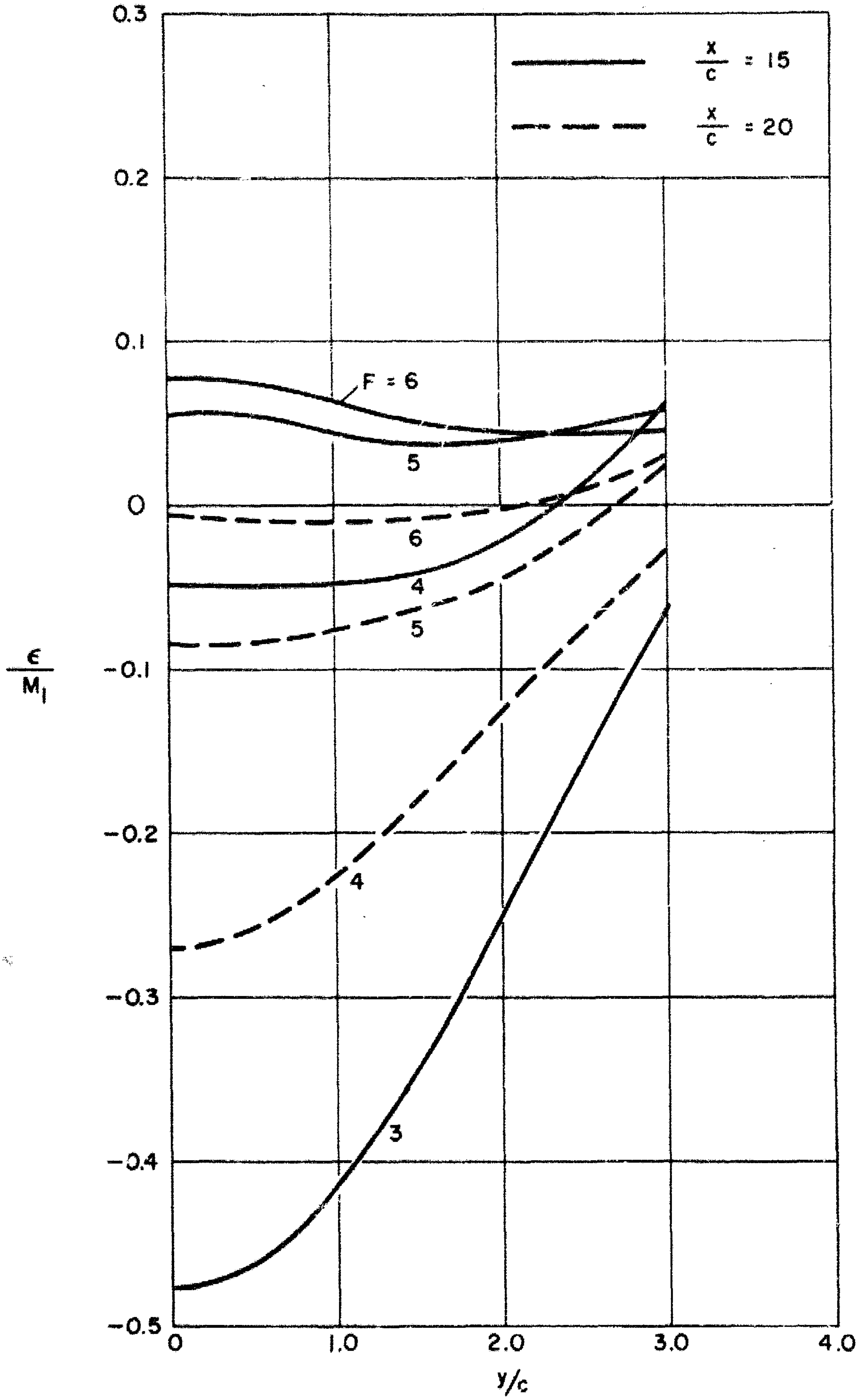



FIGURE 10— DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$AR = 6, \; f/c = 1, \; \xi_1 = 6$

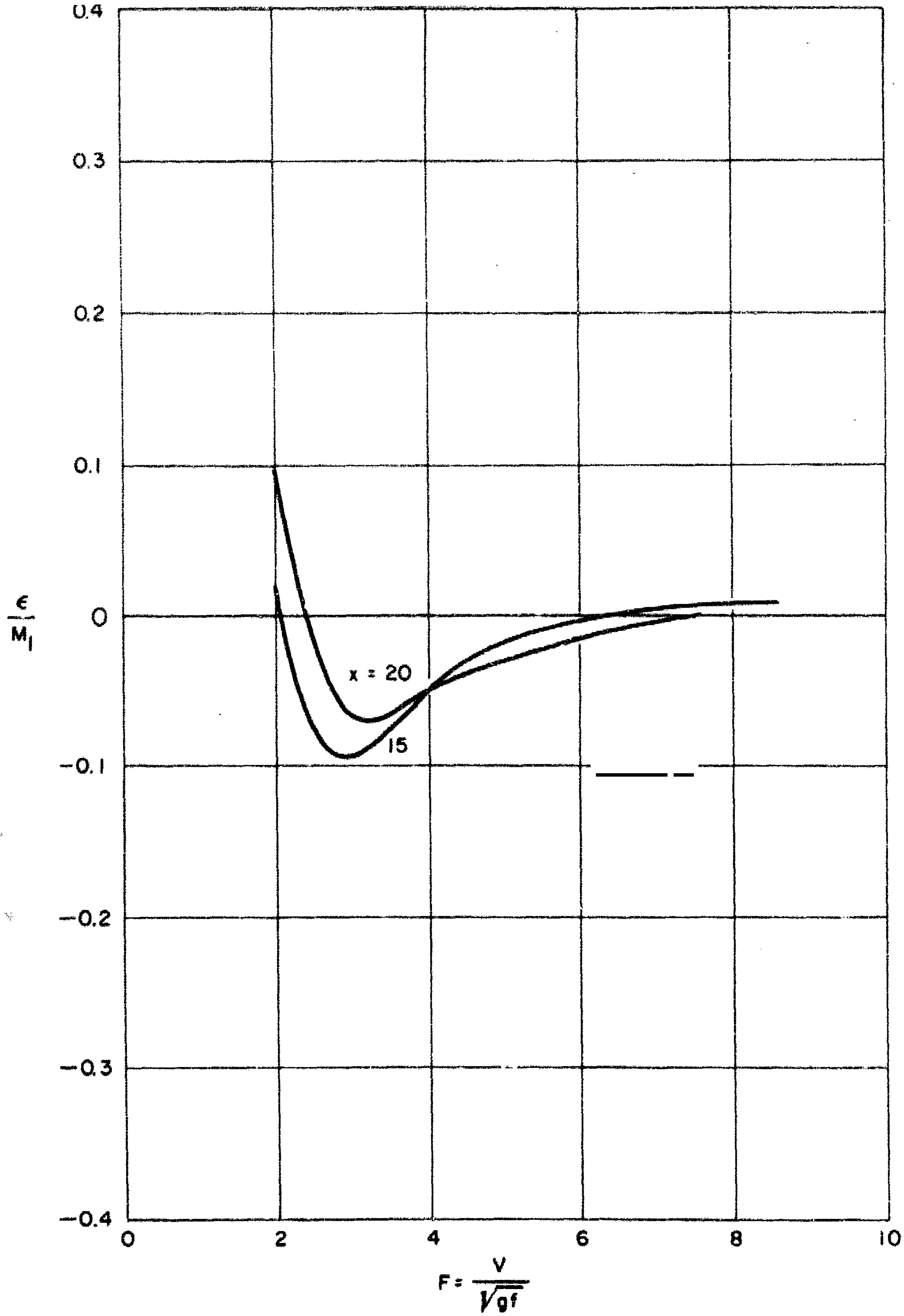
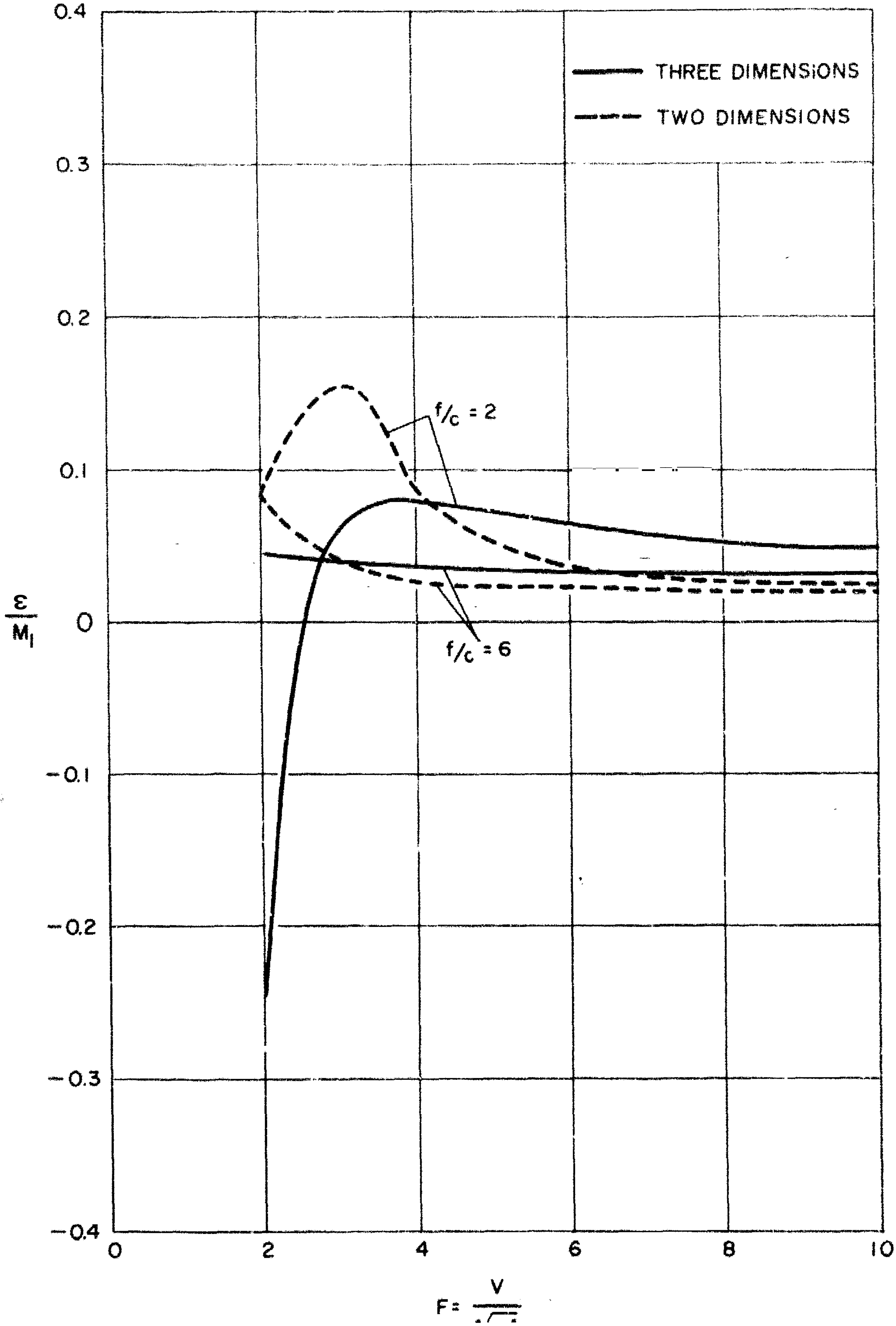



FIGURE 11 – AVERAGED DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$AR = 3,\ f/c = 1,\ \xi_1 = 2$


THREE DIMENSIONS
TWO DIMENSIONS
f/c = 2
f/c = 6
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
ε/M1
0
2
4
6
8
10
F = V

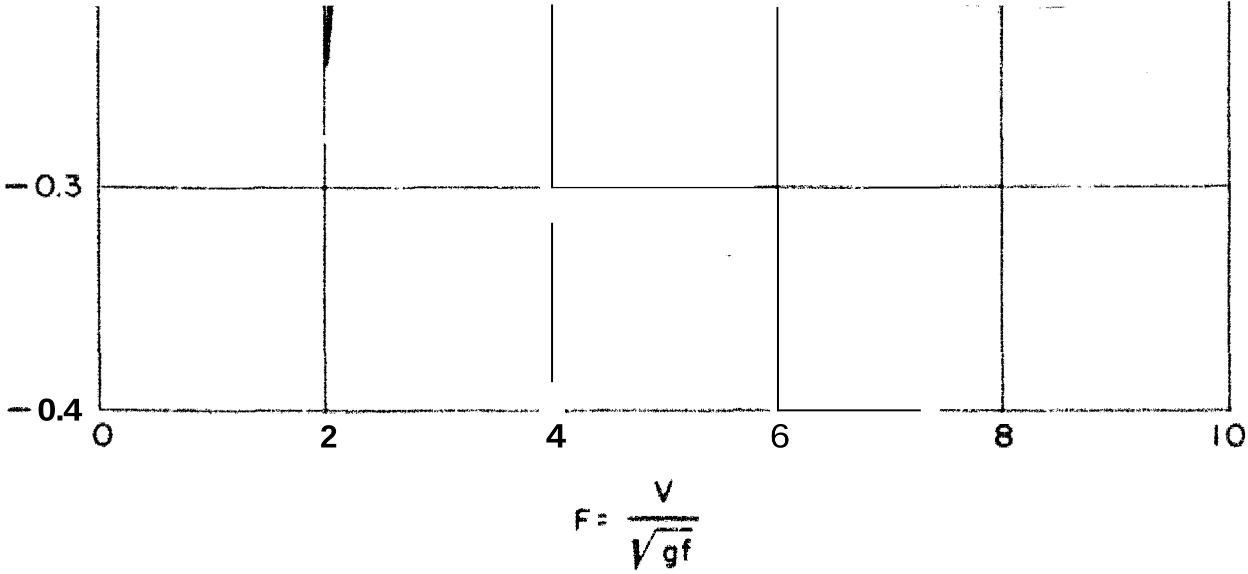



FIGURE 12 - AVERAGED DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$AR = 6, \; x/c = 15, \; \xi_1 = 6$

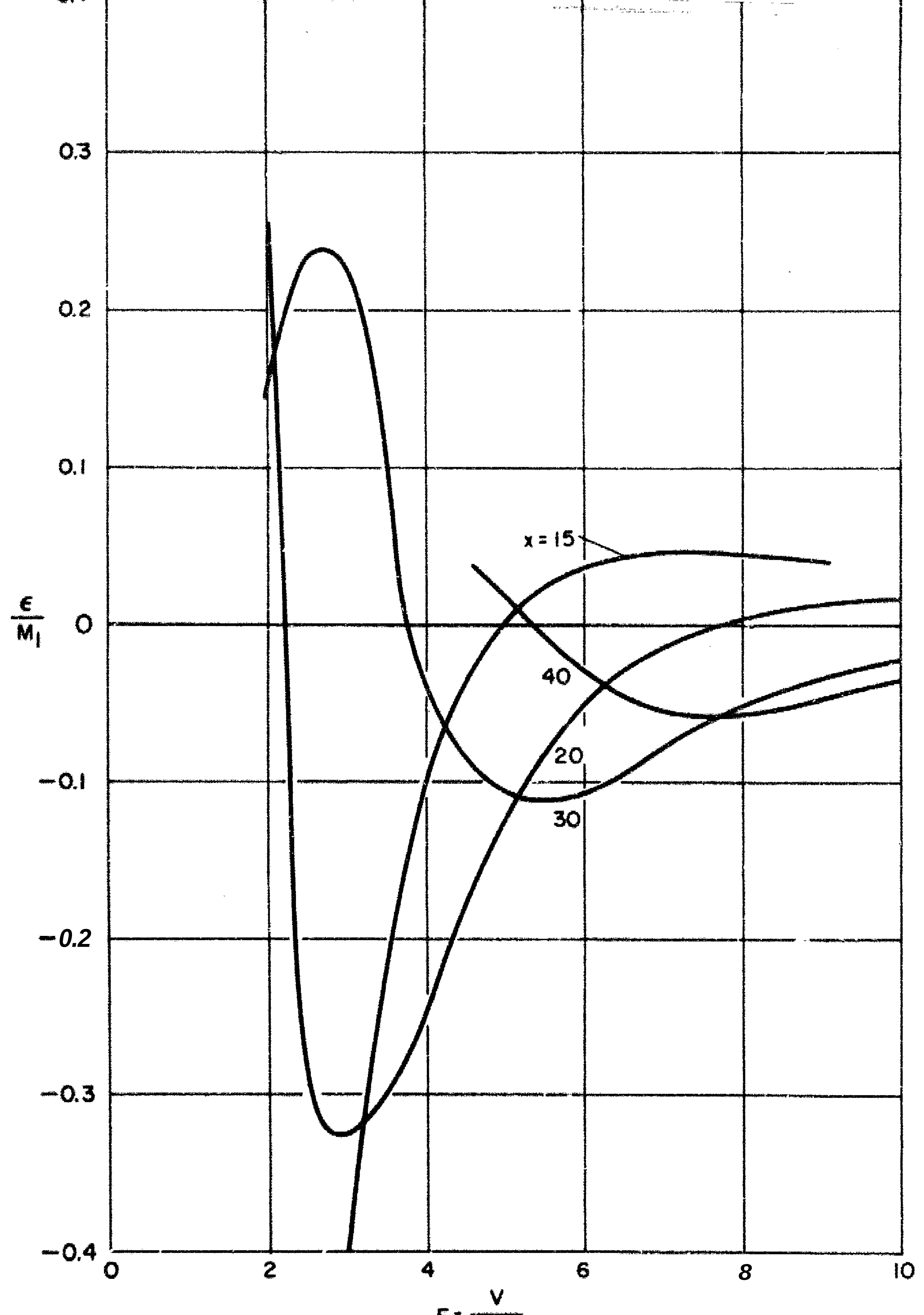



FIGURE 13 – AVERAGED DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$AR = 3, \; f/c = 1, \; \xi_1 = 8$

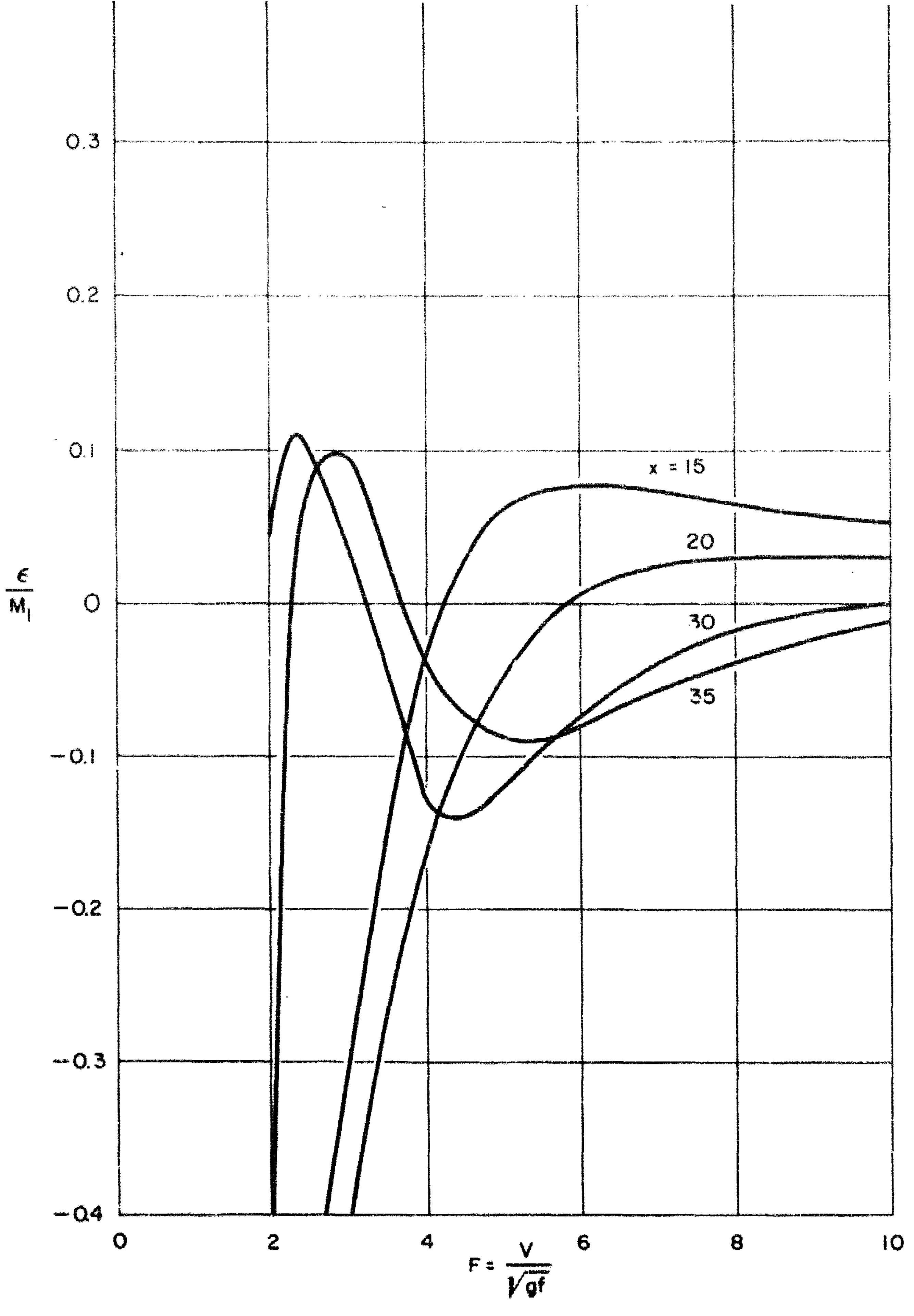
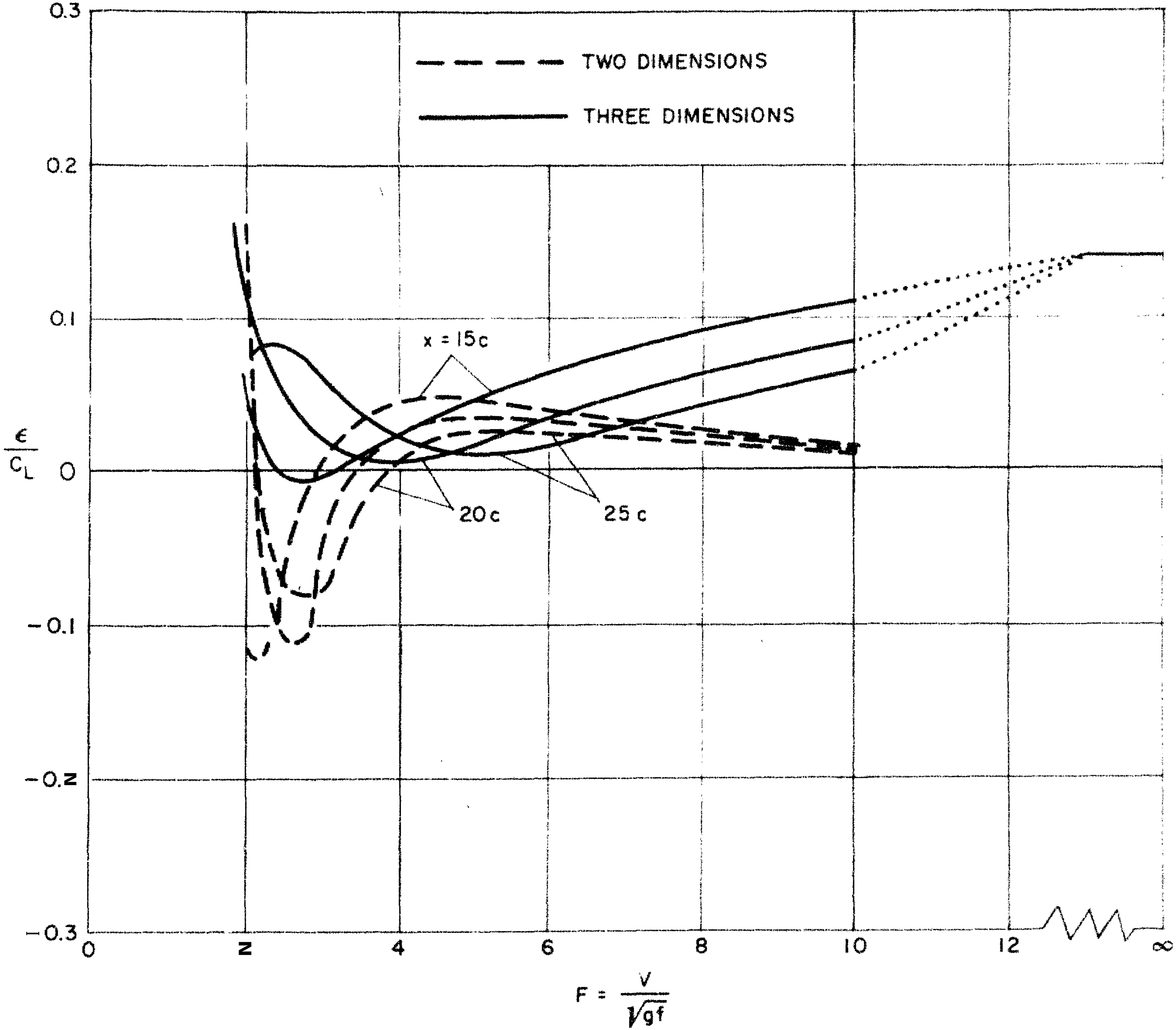



FIGURE 14 – AVERAGED DOWN-WASH DUE TO CAVITY UNDER FREE SURFACE

$AR = 6, \; f/c = 1, \; \xi_1 = 6$


TWO DIMENSIONS
THREE DIMENSIONS
x = 15c
20c
25c
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
$\frac{\epsilon}{C_L}$
0
2
4
6
8
10
12
∞
$F = \frac{V}{\sqrt{gf}}$

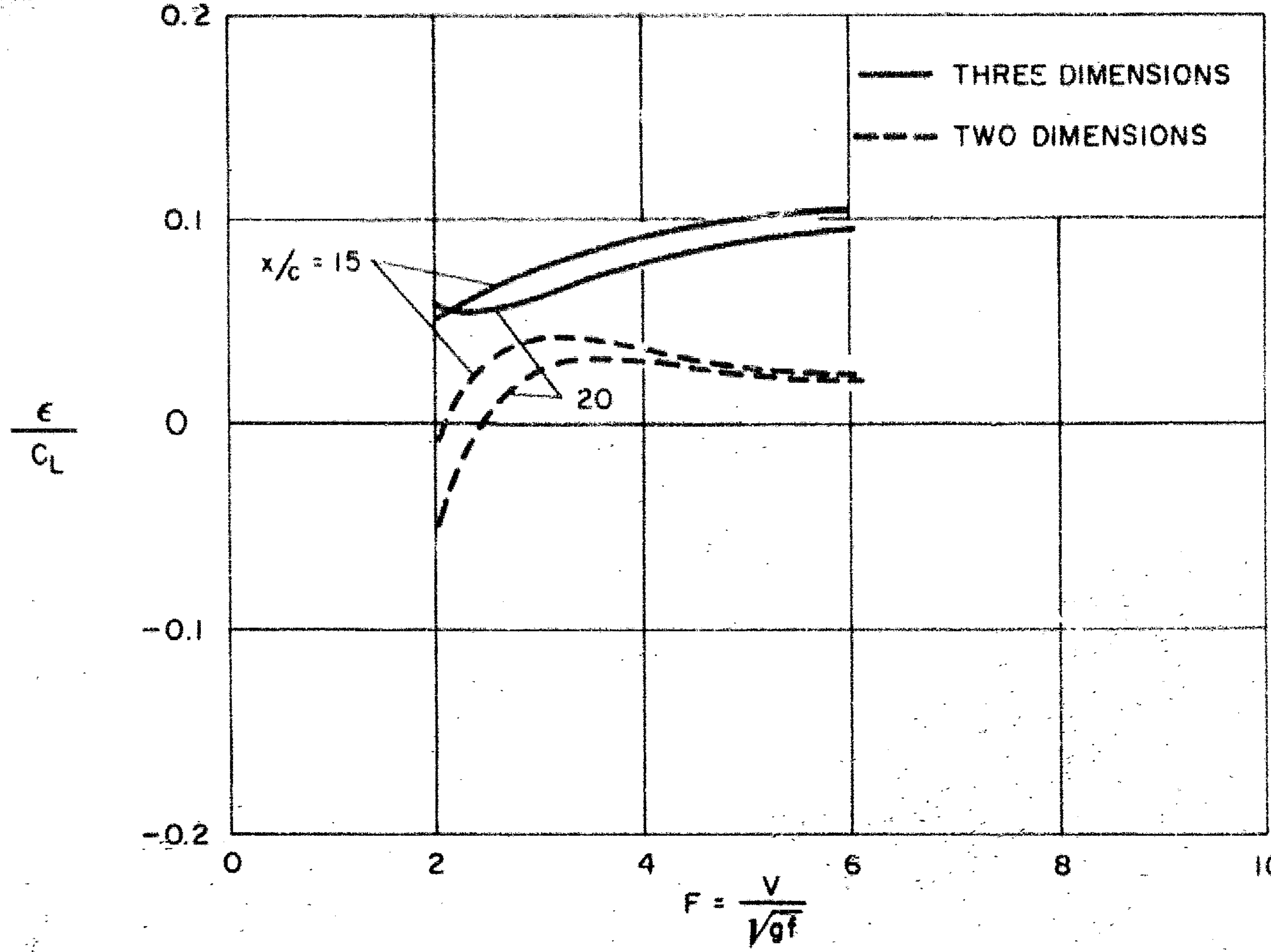



FIGURE 16 — AVERAGED DOWN-WASH DUE TO LIFTING FOIL UNDER FREE SURFACE
$AR = 6$, $f/c = 2$

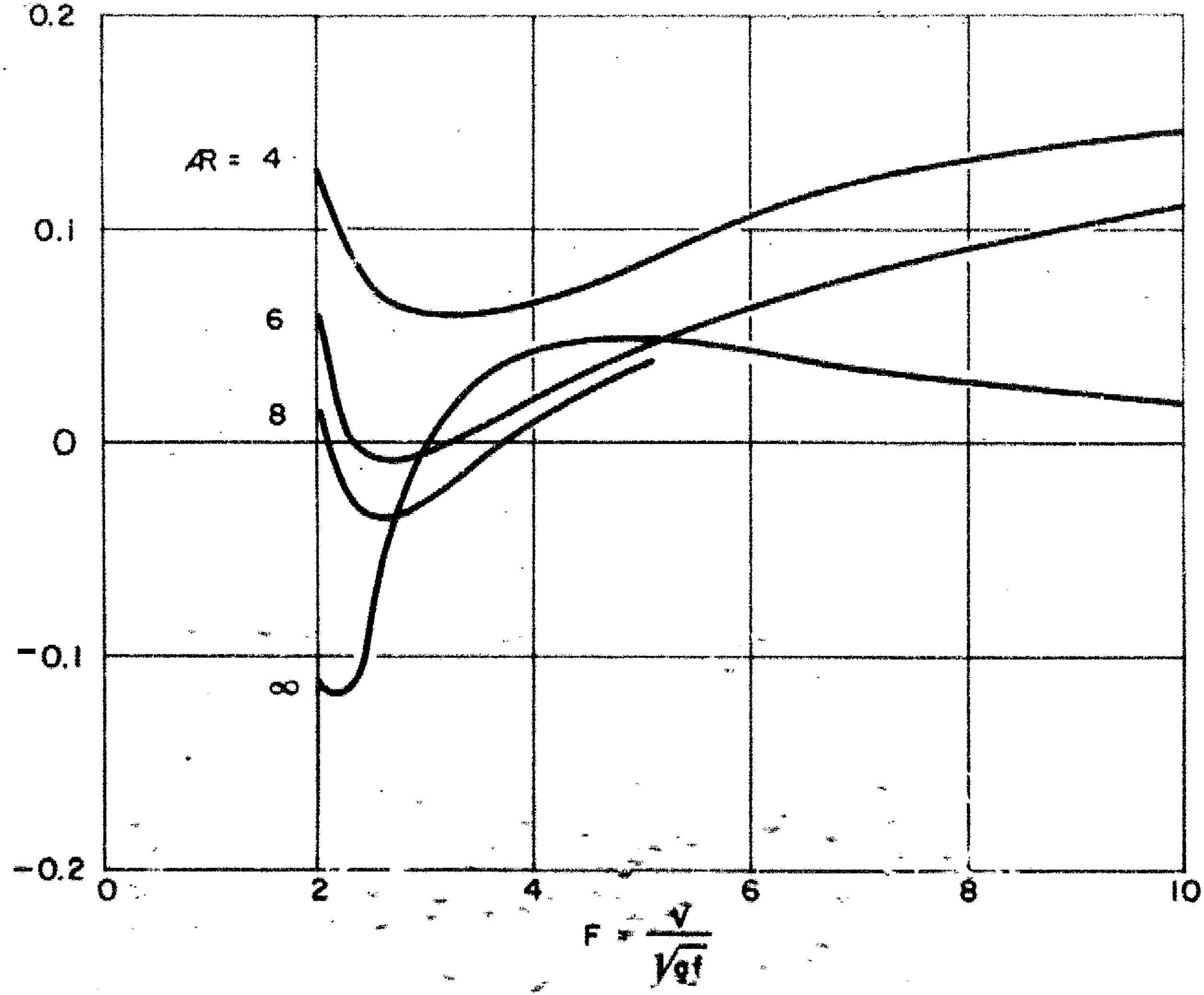



FIGURE 17 – AVERAGED DOWN-WASH DUE TO LIFTING FOIL UNDER FREE SURFACE
$f/c = 1, \; x/c = 15$